CHAPTER 4

DETERMINATION OF SATELLITE ORBITS

It has been pointed out in the preceding chapter that the equations of motion describing the two-body problem can be expressed in several ways, and several methods are available for solving these equations. But in each case a number of parameters, whose values will specify some particular orbit, appear in the results in the form of constants of integration. The actual geometric or physical meaning of these parameters, or elements of the orbit, will depend upon the method of approach used in solving the Kepler problem.

401. FORMS OF THE CONSTANTS OF INTEGRATION

Though the exact number of elements necessary is sometimes confused by a particular approach, six constants will always be needed to determine a specific conic section in space and locate the satellite body within this orbit. However, just any six might not be sufficient. This fact is in agreement with the well-known result of classical mechanics: if the force laws which govern the motion of a particle are known exactly, and if the position and velocity of the particle are known at any instant of time, as shown in figure 401A, then the motion of the particle is completely defined for all time, prior and subsequent. Since both position and velocity are vector quantities, then the knowledge of each implies the knowledge of three bits of information, in three-dimensional space at least, so six parameters are thus given.

Rather than specify some initial position and velocity vector for the satellite, one can just as well give the energy and angular momentum of the body and its initial position within the plane of its orbit (two coordinates). This may appear at first to be only four quantities, but it should be remembered that angular momentum is a vector quantity; so its three components must be given, and again six quantities result. This is illustrated in figure 401B.

Another alternative set of orbital elements is formed by the three-direction cosines of the
orbital plane, the initial position of the satellite within this plane, and the first derivatives with respect to time of these two coordinates. If this appears to be seven quantities instead of six, it should be remembered that the three-direction cosines satisfy the condition that the sum of their squares be unity; thus when two are given the third is redundant. This set is shown in figure 401C.

While each of the sets of six parameters described above are sufficient to define an unperturbed conical orbit in space, and locate the body in this orbit, none of these are commonly used. However, they can be converted into the preferred set of elements, and this conversion process is the heart of the problem. The preference for the set which will now be described is based on convenience and tradition—the astronomers have been doing it for years.

402. PREFERRED ORBITAL ELEMENTS

Two quantities specify a given conic, as can be seen from the general equation

$$r = \frac{p}{1 + e \cos \theta}$$

where the semiparameter or semilatus rectum $p$ determines the size and the eccentricity $e$ determines the general shape of the curve. The variables $r$ and $\theta$ are focal polar coordinates with origin at the central body; $r$ is called the radius and $\theta$ the true anomaly. If we restrict ourselves to elliptical orbits, it is more common to give the semimajor axis $a$ and the eccentricity. One could give the semiminor axis $b$, or the center to focus distance $c$ instead of the eccentricity, for these are all related by the equations

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}.$$  

Two angles are necessary to specify the orientation of the orbital plane to some reference $XYZ$ coordinate system with origin at the central body. The preferred elements are the dihedral angle between the orbital plane and the $XY$ plane, called the inclination $i$, and the angle in the $XY$ plane from the positive $x$ axis around to the line of intersection of the two planes where the satellite rises above the $XY$ plane. This line of intersection of the two planes is called the line of nodes, while the angle is called, in general, the longitude of the ascending node $\Omega$.

A third angle is necessary to fix the orientation of the orbit within the orbital plane. Since the periapsis, or point of closest approach of the satellite to its principal focus, lies on the major axis of the conic, or along the direction for which the true anomaly $\theta = 0$, it is convenient to use this as the reference point.

The angle measured in the orbital plane from the ascending node up to this periapsis point, in the direction of the satellite’s motion, uniquely fixes the orientation of the conic finally to the reference $XYZ$ system. It is usually called the argument of periapsis $\omega$. (Note: Periapsis is the general name. If the central body is the Sun, the term perihelion is used; if it is the Earth, then perigee.) It is convenient at this point to define another angle used later in this chapter. The sum of the argument of periapsis and true anomaly is the angle measured in the orbital plane from the ascending node up to the actual position of the satellite, in the direction of its motion, and it is called generally the argument of the latitude $\mu$. All of these angles are illustrated in figure 402.

Finally a sixth parameter is needed to establish the position of the body in its orbit at some time, or epoch. One can either give the true anomaly $\theta$ at the specified time, or more convenient, one can give the exact time the body was at the periapsis where $\theta = 0$. This choice simplifies one equation slightly, as we shall see later.

In brief summary, these are the preferred forms of the six orbital elements for any Keplerian orbit about any central body:
from this observational data the angular coordinates of the satellite relative to the center of the Earth, or to its own central body if different from the Earth. Obviously then, three such measurements of two quantities each will be necessary to determine the six elements of the orbit. If the observations are carefully selected to avoid double solutions, three will be sufficient to determine an unperturbed conical orbit. If the individual measurements can be made very accurately, all three observations can be taken while the satellite traverses only a very small portion of its orbit.

404. THE COMPUTATION OF ORBITS

The actual computation of the orbital elements, which is a basic problem in positional astronomy, can be divided into two major areas—the first orbit determination and the orbit improvements. The first determination is merely the calculation of the six elements from a minimal set of data, as described above, for a completely unknown orbit. Orbit improvements consider the data from many observations of the body, and seek to refine the elements of the first determination such that the analytical orbit fits the observed orbit as closely as possible.

The problem of a first orbit determination from only three observations was first solved by Newton for the special case of a comet travelling in a parabolic path, using graphical methods. The first complete solution by analytical methods was given by Euler in 1744. Lagrange wrote several papers on orbit theory, but his methods are not used in practice today. Laplace published a new method in 1780 which is the basis for many of the present day orbit determinations. Finally the great German mathematician Gauss conceived of another method at the age of 24, given impetus by the discovery and subsequent loss of the first minor planet, Ceres, in 1801.

The classic methods of Gauss and Laplace will now be described briefly. Inasmuch as both methods were intended for the determination of orbital elements for planets of the Sun as observed from the Earth, they necessarily involve some coordinate transformations, and assume a knowledge of the Sun's location relative to the Earth at the times of the observations. Obviously one must, in the final analysis, know some distance in order to determine an actual distance from earth to satellite, for example. If only angular quantities are known, then all distances can be found only relative to each other. In actual practice of determining orbits of the Sun's planets, the reference length is the mean distance

403. REQUIREMENTS FOR DETERMINING ELEMENTS

Let us now consider the type and number of observations of the satellite necessary and sufficient to determine all the orbital elements. By the ordinary methods of positional astronomy, two angular coordinates defining the line of sight from the observer to the body are obtained with each observation. The specific coordinate systems used, and the transformations one to another, will be considered in the following chapter. Suffice it to say that one can determine

![Diagram of orbit elements](image)
of the Earth from the Sun, defined as one astronomical unit. Its value in terms of Earth-bound standards of length such as the meter is known only approximately. Later we shall see that the problem of orbit determination is considerably simplified when we restrict ourselves to satellites of the Earth as observed from the Earth.

405. THE METHOD OF LAPLACE

The Laplacian method centers around the differential equations of motion of the satellite about its central body, and seeks to determine the position and velocity vectors of the satellite at one instant of time. This information is then transformed into the standard orbital elements. For each observation the two angular coordinates are reduced by standard techniques to qualities we shall call only the latitude $b$ and longitude $\ell$.

Other names will be applied in the following chapter where we shall consider some specific coordinate systems. From these two quantities the direction cosines $\lambda$, $\mu$, $\nu$ are determined using the equations

$$
\lambda = \cos b \cos \ell
$$

$$
\mu = \cos b \sin \ell
$$

$$
\nu = \sin b.
$$

The rectangular coordinates $x$, $y$, $z$ of the Sun with respect to the Earth are usually obtained from the American Ephemeris and Nautical Almanac.

In the first step the values of the first and second time derivatives of $\lambda$, $\mu$, $\nu$, $x$, $y$, $z$ are determined at a time near to one of the observations, say at the middle one of the three. The second step imposes the conditions that the unknown body follow the equations of motion in its path around the Sun, and likewise around the Earth. Each case is assumed to be a two-body problem, the attractions of other bodies being ignored. The result is a set of equations relating the unknown distance of the body from the Earth, its first and second time derivatives, and the unknown distance of the body from the Sun to the known quantities $\lambda$, $\mu$, $\nu$, $x$, $y$, $z$.

The third step is to determine these unknown distances from the set of equations in the preceding step and from the additional condition that the Earth, Sun, and unknown form a triangle. The result is the coordinates of the unknown body relative to the Sun. In the following step the three components of the velocity of the unknown body are determined.

The last step is to determine the elements of the orbit from a knowledge of the position and velocity vectors. This completes the brief summary\(^3\) of Laplace's method of orbit determination.

406. THE METHOD OF GAUSS

The Gaussian method can be described as based upon the integral of the equations of motion, and similarly requires only two polar coordinates for three observations, the times, and the positions of the Sun relative to the Earth. The general plan is to determine three coordinates of the unknown body at two times, from which the orbital elements can be determined using equations Gauss developed.

The first step requires that the unknown body move in a plane containing the center of the Sun. The next step imposes the condition that the body follow the law of gravitation, or that the sectors of the body's orbit between the first and second, and between the second and third radius vectors, have areas in accordance with Kepler's second law. This is done by integrating the equations of motion as a power series in the time intervals.

In the third step equations are developed to determine the distance of the unknown body from the Earth at each of the times of observation. Again the triangle condition is used to relate the distances of the unknown body from Earth and Sun to the known value of Earth-Sun distance. The next step consists of solving these equations and finding the three-dimensional Sun-centered coordinates of the body at two of the times of observation.

The final step involves the actual determination of the set of orbital elements from these two positions in the orbit. Obviously, if one were given this information at the outset, all of the lengthy procedures described above could be eliminated. In chapters 6 and 7 methods will be illustrated which give all three coordinates of an earth satellite. Anticipating the ability to find this information for two positions of the satellite, let us now consider in detail this final step of the Gaussian method, so that we will be able to determine the orbital elements of our earth satellite when the need arises.

407. SOLUTION FOR ELEMENTS GIVEN THREE-DIMENSIONAL POSITIONS

We will assume that three-dimensional fixes are given for the satellite at two instants of time, in the form of two angular coordinates
and the radial distance from the center of attraction. For the sake of generality we still call the angular coordinates simply latitude $b$ and longitude $l$. Where $C_1$ and $C_2$ are the known positions of the body at the times $t_1$ and $t_2$, the angular coordinates are as shown in figure 407A.

The inclination angle and longitude of the ascending node can be determined immediately. It follows from the spherical triangles $C_1 \Omega \ell_1$ and $C_2 \Omega \ell_2$ that

\[
\tan i \sin (\ell_1 - \Omega) = \tan b_1,
\]

\[
\tan i \sin (\ell_2 - \Omega) = \tan b_2.
\]

Since $\ell_2 - \Omega = (\ell_2 - \ell_1) + (\ell_1 - \Omega)$, the equations can be rewritten

\[
\tan i \sin (\ell_1 - \Omega) = \tan b_1,
\]

\[
\tan i \cos (\ell_1 - \Omega) = \frac{\tan b_2 - \tan b_1 \cos (\ell_2 - \ell_1)}{\sin (\ell_2 - \ell_1)}.
\]

which determine $i$ and $\Omega$ uniquely. The inclination is less or greater than $90^\circ$ according as $\ell_2$ is greater or less than $\ell_1$.

The angle along the orbit from the node to the body, or the argument of the latitude $u$, also can be obtained from the information given. It follows from the figure above that

\[
\cos (\ell_j - \Omega) \cos b_j = \cos u_j,
\]

\[
\sin (\ell_j - \Omega) \cos b_j = \sin u_j \cos i,
\]

\[
\sin b_j = \sin u_j \sin i, \quad (j = 1, 2)
\]

which uniquely define $u_1$ and $u_2$. It should be noted that all of these angles could be determined most easily by spherographical methods on a globe such as that of McMillen, if very precise answers are not as important as speed and convenience. One only need plot the two positions, given the angular coordinates, and connect them with a great circle extended through the reference or equatorial plane. A spherical protractor is necessary then to measure the inclination, the longitude of the node can be read directly if the equator is scaled, and the arguments of the latitude can be measured by appropriate dividers.

There remain the other four orbital elements to be determined from the given data, and to complete the task we must use some rather involved equations derived from Gauss. Only the results of that derivation will be given here.

Let us define the ratio of the area of the sector to that of the triangle between the two radius vectors $r_1$ and $r_2$ as $\eta$.

\[
\eta = \frac{\text{Sector}}{\text{Triangle}} = \frac{k \sqrt{p} (t_2 - t_1)}{r_1 r_2 \sin (u_1 - u_2)}
\]

where $p$ is the semiparameter of the conic, and $k$ is the force constant appearing in the equations of motion. Further, let

\[
2f = u_2 - u_1,
\]

\[
\Delta = \frac{k (t_2 - t_1)}{(2 r_1 r_2 \cos f)^{3/2}}
\]

\[
L = \frac{r_1 + r_2}{4 r_1 r_2 \cos f} - \frac{1}{2},
\]

and

\[
2g = E_2 - E_1
\]

where the $E$'s are the eccentric anomalies. Unlike true anomaly, an eccentric anomaly of an ellipse is measured at the center of the ellipse, and is defined as the angle from periapsis to the intersection of a line drawn through the satellite perpendicular to the major axis with a circle which has the major axis as a diameter. Reference to figure 407B will clarify this definition.

The two equations developed by Gauss can then be written as

\[
\eta^2 = \frac{a^2}{L + \sin^2 \frac{1}{2} g}
\]

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Figure 407B.—Relation between True and Eccentric Anomaly.

\[
\frac{n^3 - \eta^2}{a^2} = \frac{2g - \sin 2g}{\sin^3 g}
\]

in which \(\eta\) and \(g\) are the unknowns. These two equations are then solved simultaneously for \(\eta\), by letting

\[
q = \sin^2 \frac{1}{2} g,
\]

\[
\xi = \frac{2}{35} q^2 + \frac{52}{1575} q^3 + \ldots,
\]

and

\[
h = \frac{8}{6} + L + \xi.
\]

By these substitutions and the elimination of \(g\) between the two Gaussian equations, a cubic equation for \(\eta\) results with the form

\[
\eta^3 - \eta^2 - h\eta - \frac{h}{q} = 0.
\]

This equation can be solved for \(\eta\) to any desired degree of accuracy by an iterative process. If \((1/2)g\) is fairly small, then \(q\) is a small quantity of the second order, and \(\xi\) is of the fourth order. In the first approximation one assumes \(\xi\) is zero and finds \(h\). This value is substituted into the cubic and the real position root for \(\eta\) is determined. From the first of the Gauss equations it follows that

\[
q = \frac{n^2}{\eta^3} - L.
\]

The first approximation for \(\eta\) is substituted in this equation to find \(q\), and this value of \(q\) is used to determine \(\xi\) from the power series above. With this approximation for \(\xi\) an improved value for \(\eta\) is then calculated, and the process is repeated to any desired accuracy. Experience shows that this method of computing the ratio of the orbit sector to the triangle converges very rapidly, even when the time interval, and hence the value of \(g\), is fairly large. Tables to facilitate the solution are given in Watson’s Theoretical Astronomy for \(\eta\) with the argument \(h\), and for \(\xi\) with the argument \(q\). These greatly reduce the computational work.

At this point the type of conic section has been determined according as \(q\) is positive, zero, or negative, the orbit will be an ellipse, a parabola, or a hyperbola. This follows from the fact that

\[
q = \sin^2 \frac{1}{2} g = \sin^2 \frac{1}{4}(E_2 - E_1),
\]

and the eccentric anomalies are real in ellipses, zero in parabolas, and imaginary in hyperbolas.

After \(g\) has been determined by this procedure, it is a simple matter to find the remaining orbital elements. The semimajor axis is determined from the equation

\[
a = \frac{r_1 + r_2 - 2\sqrt{r_1 r_2 \cos g \cos f}}{2\sin^2 g}.
\]

The semiparameter \(p\) is given by rearranging the equation defining \(\eta\),

\[
p = \left[\frac{\eta r_1 r_2 \sin 2f}{k(t_2 - t_1)}\right]^2,
\]

and hence the eccentricity of the conic can be found using

\[
p = a(1 - e^2) \quad \text{for an ellipse, or}
\]

\[
p = a(e^2 - 1) \quad \text{for a hyperbola.}
\]

\((e = 1 \text{ for a parabola.})\)

The argument of the latitude \(u\), measured from the ascending node along the orbit to the body, has been determined above at each of times of observation. The true anomaly of the body at either of the times can be computed from the familiar polar equation of the conic,
\[ r = \frac{p}{1 + e \cos \vartheta}, \]

and since this angle originates from the periapsis point we can easily determine the argument of the periapsis by

\[ \omega = u - \theta. \]

Finally, the time of periapsis passage \( t_p \) can be found from several equations,\(^2,3\) depending upon the type of conic section.

**Parabola:**

\[ k(t - t_p) = \frac{1}{2} p^{3/2} \left[ \tan \frac{1}{2} \vartheta - \frac{1}{3} \tan \frac{1}{2} \vartheta \right]. \]

**Ellipse:**

Eccentric anomaly is determined from

\[ \tan E = \frac{\sqrt{1 - e^2} \sin \vartheta}{e + \cos \vartheta}, \]

after which Kepler's equation

\[ E - e \sin E = \frac{2 \pi}{T} (t - t_p) = \frac{\sqrt{K}}{a^{3/2}} (t - t_p), \]

yields the value of \( t_p \). \( T \) is the anomalistic period of the body.

**Hyperbola:**

The parameter \( F \) is defined by

\[ \tanh \frac{1}{2} F = \sqrt{\frac{e - 1}{e + 1}} \tan \frac{1}{2} \vartheta \]

after which \( t_p \) is given by

\[ -F + e \sinh F = \frac{k \sqrt{1 + \frac{1}{e^2}}}{a^{3/2}} (t - t_p). \]

This completes our brief survey of the methods for determining orbital parameters of a satellite from observed positional data. The reverse problem of predicting positions of a satellite at some particular time, given its orbital elements, is another of the basic problems of positional astronomy. The technique of computing an ephemeris, as such a table of predicted coordinates is called, will be considered in chapter 8.

**REFERENCES**