Symmetries + Killing Vectors

Because we can not exactly model the entire universe we approximate the geometry around a specified area

Symmetries allow us to perform these approximations

Symmetry - geometry is invariant under certain transformations that maps M to itself (isometries)

Minkowski Example

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

isometries: translations $x^\mu \to x^\mu + a^\mu$

Lorentz trans. $x^\mu \to \Lambda^\mu{}_\nu x^\nu$

Wherever $\partial_{\sigma_*} g_{\mu\nu} = 0$ - there is a symmetry under translations along x^{σ_*}

$$\partial_{\sigma_*} g_{\mu\nu} = 0 \Rightarrow x^{\sigma_*} \to x^{\sigma_*} + a^{\sigma_*}$$

Since the geodesic eqn can be written as

$$p^\lambda \nabla_\lambda p^\mu = 0$$

$$p^\lambda \partial_\lambda p_\mu - \Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = 0$$

$$p^\lambda \partial_\lambda p_\mu = m \frac{dx^\lambda}{d\tau} \partial_\lambda p_\mu = m \frac{dp_\mu}{d\tau}$$ how momentum changes on a time like path

$$\Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = \frac{1}{2} g^{\sigma\nu} (\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}) p^\lambda p_\sigma$$

$$= \frac{1}{2} (\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}) p^\lambda p^\nu$$

since $\lambda \leftrightarrow \nu$ can be interchanged in the last term $\partial_\lambda g_{\mu\nu} = \partial_\nu g_{\mu\lambda}$

$$\Rightarrow \Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = \frac{1}{2} \partial_\mu g_{\nu\lambda} p^\lambda p^\nu$$

$$\therefore \quad m \frac{dp_\mu}{d\tau} = \frac{1}{2} \partial_\mu g_{\nu\lambda} p^\lambda p^\nu$$

given an isometry $\partial_{\sigma_*} g_{\mu\nu} = 0$

$$\Rightarrow \frac{dp_{\sigma_*}}{d\tau} = 0$$ but how to find σ_* for complex systems?

if x^{σ_*} is the coord which $g_{\mu\nu}$ is ind of we can use

$$K = \partial_{\sigma_*} \Rightarrow K^\mu = (\partial_{\sigma_*})^\mu = \delta^\mu_{\sigma_*}$$

generates isometry

$$\Rightarrow p_{\sigma_*} = K^\nu p_\nu = K_\nu p^\nu$$

$$\therefore \frac{dp_{\sigma}}{d\tau} = 0 \leftrightarrow p^\mu \nabla_\mu (K_\nu p^\nu) = 0$$

$$p^\mu \nabla_\mu (K_\nu p^\nu) = p^\mu K_\nu \nabla_\mu p^\nu + p^\mu p^\nu \nabla_\mu K_\nu$$

since $p^\mu \nabla_\mu p^\nu = 0$

$$\Rightarrow p^\mu \nabla_\mu (K_\nu p^\nu) = p^\mu p^\nu \nabla_\mu K_\nu$$

$$= p^\mu p^\nu \nabla_{(\mu} K_{\nu)}$$

$$\therefore \nabla_{(\mu} K_{\nu)} = 0 \quad \text{— Killings eqn}$$

K^μ - Killing vector fields

Every Killing vector implies the existence of conserved quantities associated w/ geodesic motion.

The metric is unchanging along the direction of the Killing vector. A particle does not feel a force in the direction of the Killing vector. Momentum is conserved.

$$\nabla_{(\mu} K_{\nu_1 \dots \nu_\ell)} = 0 \text{ or } p^\mu \nabla_\mu (K_{\nu_1 \dots \nu_\ell} p^{\nu_1 \dots \nu_\ell}) = 0$$

$K_{\nu_1 \dots \nu_\ell}$ - Killing Tensor

Examples of Killing tensors are the metric and symmetrized tensor products of Killing vectors.

$$\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$$ - Homework problem

$$\Rightarrow \nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\mu} K^\nu$$

$$K^\lambda \nabla_\lambda R = 0$$ - from the Bianchi identity and Killing's eqn

geometry is not changing along a Killing vector

$$J^\mu_T = K_\nu T^{\mu\nu}$$

current conserved energy-momentum

$$\nabla_\mu J^\mu_T = \cancel{(\nabla_\mu K_\nu)}^{0} T^{\mu\nu} + K_\nu \cancel{(\nabla_\mu T^{\mu\nu})}^{0}$$

$$= 0$$

if K_ν is timelike

$$E_T = \int_\Sigma J^\mu_T n_\mu \sqrt{\gamma}\, d^3x$$

Σ ← any space-like hypersurface

total energy (conserved)

When there is a timelike Killing vector, we can write the metric ind of the timelike coord + therefore energy is conserved

Also, spacelike Killing vectors can be used to construct conserved momenta

Examples:

$$\mathbb{R}^3 \rightarrow ds^2 = dx^2 + dy^2 + dz^2$$

Killing Vectors $\begin{cases} X^\mu = (1, 0, 0) \\ Y^\mu = (0, 1, 0) \\ Z^\mu = (0, 0, 1) \end{cases}$ ← Translations

rotational symmetries

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\mathbb{R}^3 \rightarrow ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 .$$

if the metric is ind of ϕ $\therefore$ $R = \partial_\phi$ - Killing Vector

$R = x\partial_y - y\partial_x$ in cart coords

$$\therefore \quad R^\mu = (-y, x, 0) = -y\partial_x + x\partial_y$$
$$S^\mu = (z, 0, -x) = z\partial_x + x\partial_z$$
$$T^\mu = (0, -z, y) = -z\partial_y + y\partial_z$$

← Rotation about z, y, x

on a 2-sphere S^2

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \qquad x^\mu \to (r, \theta, \phi)$$

$$R = \partial_\phi$$
$$S = \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi \quad \leftarrow \text{rotations}$$
$$T = -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\phi$$

Killing vector fields may be linearly ind

lin comb of Killing vectors are Killing vectors
if multipled by constant coefficients

Maximally Symmetric Spaces

$\mathbb{R}^n$ flat Euclidean metric - highest possible degree of symmetry, isometries consist of translations and Rotations transformations in n-dimensions

n - total translations

$\frac{1}{2}n(n-1)$ - independent rotations

$\therefore\ n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ ind symmetries

This works for a manifold which looks locally like $\mathbb{R}^n$ flat Euclidean space

# of isometries = # of linearly ind Killing vector fields

∴ $\frac{1}{2}n(n+1)$-Killing vectors is maximally symmetric space

maximally symmetric implies that the curvature is the same everywhere and the same in every direction
(translations) (rotations)

Use locally inertial coords at some pt p, $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$

$\eta_{\hat{\mu}\hat{\nu}} = \Lambda \eta_{\hat{\mu}\hat{\nu}}$ – the metric is invariant under Lorentz transformations

$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$ – is therefore also Lorentz invariant

$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \propto g_{\hat{\rho}\hat{\mu}} g_{\hat{\sigma}\hat{\nu}} - g_{\hat{\rho}\hat{\nu}} g_{\hat{\sigma}\hat{\mu}}$ — this is true in any coord system at any point

$$\Rightarrow R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})$$

R - represents the classification + scaling of the maximally symmetric space

$R > 0$ - spheres

$R = 0$ - **plane**

$R < 0$ - hyperboloids

H^n - hyperboloids - Maximally symmetric Euclidean spaces w/ negative curvature

Poincaré half-plane: H^2

for $y > 0$ $\qquad ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2)$

$$\Delta s = \int_{y_1}^{y_2} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dy}\frac{dx^\nu}{dy}}\, dy = \int_{y_1}^{y_2} \sqrt{\frac{a^2}{y^2} 1 \cdot 1}\, dy$$

(for x = const)

$$= a\int_{y_1}^{y_2} \frac{dy}{y}$$

$$= a \ln\left(\frac{y_2}{y_1}\right)$$

in Flat Euclidean space $\quad \Delta s(\text{for } x = \text{const}) = y_2 - y_1$

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -y^{-1}$$

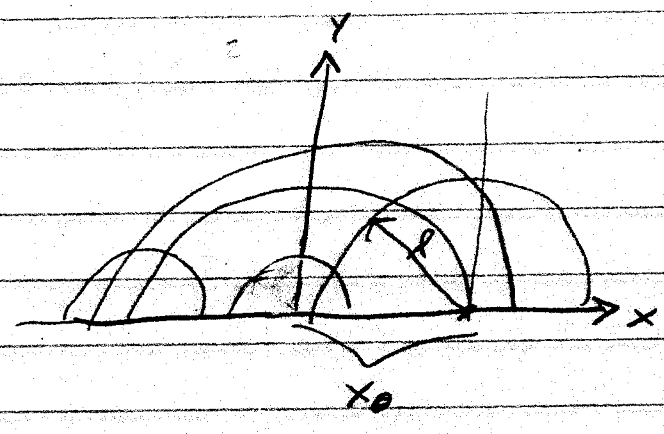
$$\Gamma^y_{xx} = y^{-1} \qquad \therefore \quad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$\Gamma^y_{yy} = -y^{-1} \qquad \Rightarrow (x - x_0)^2 + y^2 = l^2$$

where x_0 + l are constants

as $x_0 \to \infty \quad l \to \infty$

and $l - x_0$ is fixed



$$R^x{}_{yxy} = -y^{-2} \Rightarrow \begin{array}{l} R_{xx} = -y^{-2} \\ R_{yy} = -y^{-2} \end{array}$$

$$R = -\frac{2}{a^2}$$

same as S^2 with a diff sign

also $R^x{}_{yxy} = -y^{-2} = \frac{1}{2(2-1)}\left(-\frac{2}{a^2}\right)\cdot\frac{y^2}{a^2}\left(\frac{a^2}{y^2}\cdot\frac{a^2}{y^2} - 0\right) = -y^{-2}$

Local Geometry vs Global Topology

Gauss-Bonnet theorem: $\chi(M) = \frac{1}{4\pi}\int_M R\sqrt{|g|}\, d^n x$

determines Global Curvature

Euler characteristic

$\chi(M) = 2(1-g)$ in 2D

if $g \geq 2$ curvature is negative

for Lorentz signatures
Maximally Symmetric Spacetimes
$$\begin{cases} R = 0 & \text{- Minkowski} \\ R > 0 & \text{- de Sitter} \\ R < 0 & \text{- anti-de Sitter} \end{cases}$$

Geodesic Deviation

In curved space "parallel" geodesics will eventually cross

consider $\gamma_s(t)$ - one parameter family of geodesics

for $s \in \mathbb{R}$, γ_s is parameterized by t

$\therefore$ a surface is created which is parameterized by $s + t$, if the geodesics don't cross

$$x^\mu(s,t) \in M \Rightarrow T^\mu = \frac{\partial x^\mu}{\partial t}, \quad S^\mu = \frac{\partial x^\mu}{\partial s}$$

points along geodesics

points to other geodesics

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S$$ — relative velocity of geodesics

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu$$ — relative acceleration of geodesics

note $[S,T]^\mu = 0 = S^\lambda \nabla_\lambda T^\mu - T^\lambda \nabla_\lambda S^\mu$

$$\therefore S^\lambda \nabla_\lambda T^\mu = T^\lambda \nabla_\lambda S^\mu$$

$$A^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu)$$

$$= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= (T^\rho \nabla_\rho S^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$$= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

$$= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + S^\sigma \nabla_\sigma \cancelto{0}{(T^\rho \nabla_\rho T^\mu)}$$

$$- (S^\sigma \nabla_\sigma T^\rho)(\nabla_\rho T^\mu) + R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

$$= R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

$$\therefore \quad A^\mu = \frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

geodesic deviation eqN

relative acceleration of geodesics is proportional to curvature

This is a manifestation of gravitational tidal forces