# The Expanding Universe Revisited

$$ds^2 = -dt^2 + a^2(t)\,[dx^2 + dy^2 + dz^2]$$

$$= -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j = g_{\mu\nu}\,dx^\mu dx^\nu$$

from $I = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau = \frac{1}{2}\int\left(\frac{ds}{d\tau}\right)^2 d\tau$

$$= \frac{1}{2}\int\left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]d\tau$$

using $x^\mu \to x^\mu + \delta x^\mu$ and $\delta I = 0$

we get $n$ eqns, one for each dimension

from $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$ we get $\Gamma^\mu_{\rho\sigma}$

for $t \to t + \delta t$

$$a(t+\delta t) = a(t) + \dot{a}\,\delta t$$

$$\therefore\ \delta I = \frac{1}{2}\int\left[-2\frac{dt}{d\tau}\frac{d(\delta t)}{d\tau} + 2a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\,\delta t\right]d\tau$$

$$= \int\left[\frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]\delta t\,d\tau$$

using IBP

$$\Rightarrow \frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0$$

$$\therefore\ \Gamma^0_{ij} = a\dot{a}\,\delta_{ij}\ ,\quad \Gamma^0_{00} = \Gamma^0_{i0} = \Gamma^0_{0i} = 0$$

like $\frac{d^2x^0}{dt^2} + \Gamma^0_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$

for $x^i \to x^i + \delta x^i$

$$\delta I = \frac{1}{2}\int a^2\left(2\,\delta_{ij}\frac{dx^i}{d\tau}\frac{d(\delta x^j)}{d\tau}\right)d\tau$$

$$= -\int\left(a^2\frac{d^2x^i}{d\tau^2} + 2a\frac{da}{d\tau}\frac{dx^i}{d\tau}\right)\delta_{ij}\,\delta x^j\,d\tau$$ IBP

$$\frac{da}{d\tau} = \dot{a}\frac{dt}{d\tau}$$

$$\therefore \frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0$$

$$\Rightarrow \Gamma^i_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau}$$

$$\Gamma^i_{00} = \Gamma^i_{jk} = 0\,,\quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$$

for null geodesics $ds^2 = 0$

$$\therefore 0 = -dt^2 + a^2(t)\,dx^2 \quad \text{— in } 1+1 \text{ dim}$$

$$\Rightarrow \frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}$$

using $\frac{d^2x^0}{d\tau^2} + \Gamma^0_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$ as $\tau \to \lambda$

$$\frac{d^2t}{d\lambda^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0$$

where $\frac{dx^i}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}$

$$\therefore \frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0 \quad \therefore \quad \frac{dt}{d\lambda} = \frac{\omega_0}{a}$$

given $a(t)$ you can find $t(\lambda)$

$$E = -p_\mu U^\mu$$

$$= -g_{00} \frac{dx^0}{d\lambda} U^0$$

$$= \frac{\omega_0}{a} \quad \text{if } U^\mu = (1,0,0,0)$$

if $a = 1$ & $\hbar = 1 \Rightarrow E = \hbar \omega_0$

cosmological redshift

photon observed

$$\frac{E_2}{E_1} = \frac{a_1}{a_2}$$

photon emitted

since wavelength is inversely proportional to freq if $a(t)$ increases w/ time the wavelength also grows w/ time

$$\text{redshift} \quad z = \frac{\omega_1 - \omega_2}{\omega_2} = \frac{a_2}{a_1} - 1$$

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| Spacetime | Flat | General |
| --- | --- | --- |
| | $\partial_\mu$ | $\nabla_\mu$ |
| | $\eta_{\mu\nu}$ | $g_{\mu\nu}$ |

Example $\partial_\mu T^{\mu\nu} = 0 \rightarrow \nabla_\mu T^{\mu\nu} = 0$

conservation of 4-momenta

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p g^{\mu\nu}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^{-2} & & \\ & & a^{-2} & \\ & & & a^{-2} \end{pmatrix}$$

$$U^\mu = (1, 0, 0, 0)$$

$$\therefore \quad T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p/a^2 & & \\ & & p/a^2 & \\ & & & p/a^2 \end{pmatrix}$$

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0$$

for $\nu = 0$: $\partial_\mu T^{\mu 0} = \partial_0 T^{00} = \dot\rho$

$$\Gamma^\mu_{\mu\lambda} T^{\lambda 0} = \Gamma^\mu_{\mu 0} T^{00} = \frac{\dot a}{a}\delta^i_i \rho = 3\frac{\dot a}{a}\rho$$

$$\Gamma^0_{\mu\lambda} T^{\mu\lambda} = \cancel{\Gamma^0_{00}}^{\,0} T^{00} + \sum_{i=1}^{3} \Gamma^0_{ii} T^{ii} = 3a\dot a \, p/a^2$$

$$= 3\frac{\dot a}{a} p$$

$$\therefore \quad \dot\rho + 3\frac{\dot a}{a}\rho + 3\frac{\dot a}{a}p = 0 \Rightarrow \dot\rho = -3\frac{\dot a}{a}(\rho + p)$$

for $\nu = i$: $\partial_\mu T^{\mu i} = \partial_i T^{ii} = \frac{1}{a^2}\partial_i p$

$$\Gamma^\mu_{\mu\lambda} T^{\lambda i} = \Gamma^\mu_{\mu i} T^{ii} = 0$$

$$\Gamma^i_{\mu\lambda} T^{\mu\lambda} = \cancelto{0}{\Gamma^i_{00}} T^{00} + \cancelto{0}{\Gamma^i_{jj}} T^{jj} = 0$$

$$\therefore \; \partial_i p = 0$$

for Minkowski $a = 1$ $\dot{a} = 0 \Rightarrow \dot{\rho} = 0, \; \partial_i p = 0$

try $p = w\rho$

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1+w) \Rightarrow \rho \propto a^{-3(1+w)}$$

dust $w = 0$ $\rightarrow p = 0, \; \rho \propto a^{-3}$

radiation $w = \frac{1}{3}$ $\rightarrow p = \frac{1}{3}\rho, \; \rho \propto a^{-4}$

vacuum $w = -1$ (Minkowski) $\rightarrow p = -\rho, \; \rho = \text{const}$

$E = \int \rho a^3 d^3x$ — const for dust

from $\sqrt{|g|}$ decreases for radiation

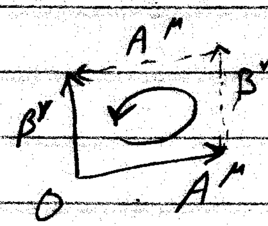
increases for vacuum

Energy is invariant b/c of the homogenarty of time if the universe changes w/ time energy is not conserved, However 4-momenta is conserved

Riemann

Flatness – parallel transport is path independent

look at parallel transport around an infinitesimal loop




$A^\mu \to B^\nu \to -A^\mu \to -B^\nu$

Some tensor should explain how the vector changes when it reaches the starting pt.

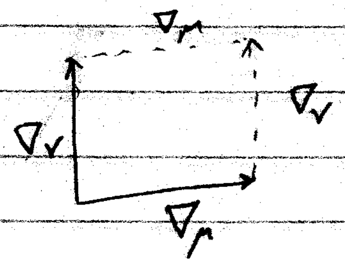
It should be linear, depend on A + B and anti symmetric in A + B so a reverse parallel transport undoes the change

parallel transported vector

$$\Rightarrow \delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$$

Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$




– replace A + B w/ covariant derivatives

The commutator of the covarient derivatives is zero in the absence of curvature

$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho$$

$$= \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} \nabla_\nu V^\sigma - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho$$

$$- \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda - \cdots$$

$$= (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}) V^\sigma$$

$$- 2 \Gamma^\lambda_{[\mu\nu]} \nabla_\lambda V^\rho$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} \nabla_\lambda V^\rho$$

where $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$

$R^\rho{}_{\sigma\mu\nu}$ - measures the part of the commutator that is prop to the vector field

$T^\lambda_{\mu\nu}$ - torsion measures the part prop to the covariant derivative of the vector field

torsion map $\rightarrow$ $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

similar to Yang-Mills eqn

Riemann tensor $\rightarrow$ $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

commutation of $\nabla$'s

where $\nabla_X = X^\mu \nabla_\mu$

$$\Rightarrow R^\rho{}_{\sigma\mu\nu} X^\mu Y^\nu Z^\sigma = X^\lambda \nabla_\lambda (Y^\eta \nabla_\eta Z^\rho) - Y^\lambda \nabla_\lambda (X^\eta \nabla_\eta Z^\rho)$$
$$- (X^\lambda \partial_\lambda Y^\eta - Y^\lambda \partial_\lambda X^\eta) \nabla_\eta Z^\rho$$

Since $R^\rho{}_{\sigma\mu\nu}$ is characterized by $\Gamma^\sigma_{\mu\nu}$, if $\Gamma^\sigma_{\mu\nu}$ is derived from the metric, $R^\rho{}_{\sigma\mu\nu}$ refers to the curvature of the metric

- If a coord system exists where the components of the metric are const, $R^\rho{}_{\sigma\mu\nu}$ will vanish

- If $R^\rho{}_{\sigma\mu\nu}$ vanishes, we can construct a coord system where the metric components are const

Properties

$n^4$ - ind components $\sim 4^4 = 256$

anti-symmetric in last 2 components implies $\frac{1}{2} n(n-1)$ for the last 2 indices
$\therefore$ only $\frac{1}{2} n^3(n-1)$ total ind components

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$$

in locally inertial coords at pt $p$ $\Gamma^{\hat\lambda}_{\hat\mu\hat\sigma} = 0$

but $\partial_{\hat\nu} \Gamma^{\hat\lambda}_{\hat\mu\hat\sigma} \overset{?}{=} 0$

$$R_{\hat\rho\hat\sigma\hat\mu\hat\nu}(p) = g_{\hat\rho\hat\lambda} \left( \partial_{\hat\mu} \Gamma^{\hat\lambda}_{\hat\nu\hat\sigma} - \partial_{\hat\nu} \Gamma^{\hat\lambda}_{\hat\mu\hat\sigma} \right)$$

$$= \frac{1}{2} \underbrace{g_{\hat\rho\hat\lambda} g^{\hat\lambda\hat\tau}}_{\delta^{\hat\tau}_{\hat\rho}} \left( \cancel{\partial_{\hat\mu}\partial_{\hat\nu} g_{\hat\sigma\hat\tau}} + \partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\tau\hat\nu} - \partial_{\hat\mu}\partial_{\hat\tau} g_{\hat\nu\hat\sigma} \right.$$

$$\left. - \cancel{\partial_{\hat\nu}\partial_{\hat\mu} g_{\hat\sigma\hat\tau}} - \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\tau\hat\mu} + \partial_{\hat\nu}\partial_{\hat\tau} g_{\hat\mu\hat\sigma} \right)$$

$$= \frac{1}{2} \left( \partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\rho\hat\nu} - \partial_{\hat\mu}\partial_{\hat\rho} g_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\rho\hat\mu} + \partial_{\hat\nu}\partial_{\hat\rho} g_{\hat\mu\hat\sigma} \right)$$

Notice $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ + $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

$$\Rightarrow R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$$

$$\therefore R_{\rho[\sigma\mu\nu]} = 0$$

b/c we can veiw $R_{\rho\sigma\mu\nu}$ as $R_{[\rho\sigma][\mu\nu]}$ there are

$$\frac{1}{2}\left[\frac{1}{2}n(n-1)\right]\left[\frac{1}{2}n(n-1)+1\right] = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n)$$

ind components

b/c of $R_{\rho[\sigma\mu\nu]} = 0 \Rightarrow R_{[\rho\sigma\mu\nu]} = 0$

if $R_{\rho\sigma\mu\nu} = X_{\rho\sigma\mu\nu} + R_{[\rho\sigma\mu\nu]}$

there are

$$\underbrace{\tfrac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n)}_{\text{from } R_{[\rho\sigma][\mu\nu]}} - \underbrace{\tfrac{1}{24}n(n-1)(n-2)(n-3)}_{\text{from } R_{[\rho\sigma\mu\nu]} = 0}$$

$$= \frac{1}{12}n^2(n^2-1) \text{ ind components}$$

$\therefore$ for $n = 4$ there are only 20 ind components

these are the 20 unknown degrees of freedom in the transformation to locally inertial coords

b/c of $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad \text{- Bianchi identity}$$

in general the torsion tensor adds more terms

$$\Rightarrow [[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho] = 0$$

To make the Riemann tensor easier to handle for physical interpretations in a coord invariant way we either break it up into symmetric + antisymmetric parts or contract indices symmetric

anti-symmetric

example $X_{\mu\nu} = X_{(\mu\nu)} + X_{[\mu\nu]}$

$$X = g^{\mu\nu} X_{(\mu\nu)} \qquad \hat{X}_{\mu\nu} = X_{(\mu\nu)} - \frac{1}{n} X g_{\mu\nu}$$

trace

trace-free

$$\therefore X_{\mu\nu} = \frac{1}{n} X g_{\mu\nu} + \hat{X}_{\mu\nu} + X_{[\mu\nu]}$$

Ricci Tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \Rightarrow R_{\mu\nu} = R_{\nu\mu}$$

using the Christoffel connection

only ind contraction

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

Ricci Scalar

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu}$$ — trace-free

Weyl tensor - captures info about the trace-free parts of the Riemann tensor

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma} R$$

All possible contractions of $C_{\rho\sigma\mu\nu}$ vanish

$$C_{\rho\sigma\mu\nu} = C_{[\rho\sigma][\mu\nu]}$$

$$C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}$$

$$C_{\rho[\sigma\mu\nu]} = 0$$

Weyl is only defined in 3 or more dimensions
" " invariant under conformal transformations

$$\therefore\ C^{\rho}{}_{\sigma\mu\nu}(g_{\mu\nu}) = C^{\rho}{}_{\sigma\mu\nu}(\omega^2(x)\, g_{\mu\nu})$$

$$0 = g^{\nu\sigma} g^{\mu\lambda} (\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu})$$

$$= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu}$$

$$\Rightarrow\ \nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R$$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ – Einstein Tensor

$$\therefore\ \nabla^\mu G_{\mu\nu} = 0$$

old thinking – curvature depends on the extrinsic geometry of the manifold, which relates the space to a larger space new thinking - Riemann curvature is a property of the intrinsic geometry of a space as measured by observers on the manifold

2-sphere ($S^2$) example

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$a \sim$ radius (of curvature)

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

all other $\Gamma = 0$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

$$R^\theta{}_{\phi\theta\phi} = \sin^2\theta$$

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda} R^\lambda{}_{\phi\theta\phi} = a^2\sin^2\theta$$

$$R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = 1$$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta$$

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}$$

$\leftarrow$ curvature decreases as radius increases