Curvature

Connection - relates vectors in the tangent spaces at nearby pts

Christoffel symbol - $\Gamma_{\mu\nu}^{\lambda}$

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} (g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})$$

where $g_{\nu\sigma,\mu} = \partial_\mu g_{\nu\sigma}$

covariant derivatives - ∇_μ

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\sigma}^{\nu} V^\sigma = V^\nu_{;\mu}$$

geodesics - parameterized curve $x^\mu(\lambda)$ which obeys

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \leftarrow \text{geodesic eqn}$$

Riemann tensor - technical expression of curvature

$$R^\rho_{\ \sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

vanishes only if there is no curvature

Covariant Derivatives

- Because of coord. transformations we can't determine curvature by simply looking at the metric

- Because the partial derivative isn't a good tensor operator, we need the connection to build the Covariant Derivative which is but reduces to a partial derivative in the absence of curvature.

- The covariant derivative should be a coord ind form of the partial derivative

Properties
1. linearity : $\nabla(T+S) = \nabla T + \nabla S$

2. Leibniz rule : $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

∇ - should be a partial derivative + some linear transform

$$\therefore \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

transformation

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$$

$$= \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'}$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda$$

when you expand $\nabla_\mu V^\nu$ on the rhs

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda$$

$$\therefore \nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda$$

$$+ \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda$$

$$\Rightarrow \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda$$

$$\therefore \Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}$$

not a tensor

for a dual vector

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}^\lambda_{\mu\nu} \omega_\lambda$$

same transformation properties as $\Gamma^\lambda_{\mu\nu}$

2 more properties of the covariant derivative

3. commutes w/ contractions $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$

4. reduces to partials on scalars $\nabla_\mu \phi = \partial_\mu \phi$

$$\nabla_\mu (\omega_\lambda V^\lambda) = (\nabla_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\nabla_\mu V^\lambda)$$

$$= (\partial_\mu \omega_\lambda) V^\lambda + \tilde{\Gamma}^\sigma_{\mu\lambda} \omega_\sigma V^\lambda + \omega_\lambda (\partial_\mu V^\lambda) + \omega_\lambda \Gamma^\lambda_{\mu\rho} V^\rho$$

$$= \partial_\mu (\omega_\lambda V^\lambda)$$

$$= (\partial_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\partial_\mu V^\lambda)$$

$$\therefore \tilde{\Gamma}^\sigma_{\mu\lambda} \omega_\sigma V^\lambda = - \omega_\lambda \Gamma^\lambda_{\mu\rho} V^\rho$$

$$\Rightarrow \tilde{\Gamma}^\sigma_{\mu\lambda} = - \Gamma^\sigma_{\mu\lambda}$$

$$\therefore \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda$$

$$\nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_l} = \partial_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_l}$$

$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_l} + \cdots$$

$$- \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_l} - \cdots$$

$\Gamma^\lambda_{\mu\nu} \rightarrow n^3$ components for $n = 4$ dim $\rightarrow n^3 = 64$

connection - transports a vector from one tangent space to another

$S^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\mu\nu}$ ← diff between connections is a tensor

$$\nabla_\mu V^\lambda - \tilde{\nabla}_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma^\lambda_{\mu\nu} V^\nu - \partial_\mu V^\lambda - \tilde{\Gamma}^\lambda_{\mu\nu} V^\nu$$

$$= S^\lambda_{\mu\nu} V^\nu$$

$$\therefore \Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + S^\lambda_{\mu\nu}$$

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]}$$

torsion tensor

2 more properties to isolate a connection/metric relationship

5. $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$ – torsion-free $\rightarrow T^\lambda_{\mu\nu} = 0$

6. $\nabla_\rho g_{\mu\nu} = 0$ – metric compatibility

$$\Rightarrow \nabla_\lambda \epsilon_{\mu\nu\rho\sigma} = 0$$

$$\nabla_\rho g^{\mu\nu} = 0$$

$$\therefore g_{\mu\lambda} \nabla_\rho V^\lambda = \nabla_\rho (g_{\mu\lambda} V^\lambda) = \nabla_\rho V_\mu$$

metric commutes with covariant derivative

$\therefore$ There is only one torsion-free metric compatible connection per manifold

3 permutations of $g_{\mu\nu}$

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0$$

$$\Rightarrow \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0$$

$$\therefore \Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

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for $ds^2 = dr^2 + r^2 d\theta^2$ - Flat space polar coords

$$g^{rr} = 1 \quad g^{\theta\theta} = \frac{1}{r^2}$$

$$\Gamma^r_{rr} = \frac{1}{2} g^{r\rho}(\partial_r g_{r\rho} + \partial_r g_{\rho r} - \partial_\rho g_{rr})$$

$$= \frac{1}{2} g^{rr}(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr})$$

$$+ \frac{1}{2} g^{r\theta}(\partial_r g_{r\theta} + \partial_r g_{\theta r} + \partial_\theta g_{rr})$$

$$= 0$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{r\rho}(\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta})$$

$$= \frac{1}{2} g^{rr}(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta})$$

$$= -r$$

$$\Rightarrow \Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0$$

$$\Gamma^\theta_{rr} = 0$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^\theta_{\theta\theta} = 0$$

- from this we get formulas for divergence, gradient + curl in curvilinear coords

- We can make the metric appear flat at a point i.e. we can make the Christoffel symbols vanish at a pt.

divergence $\rightarrow$ $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}$$ $\leftarrow$ show on homework #3

$$\therefore \nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|}\, V^\mu)$$

Stoke's theorem in curved space

$$\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|}\, d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|}\, d^{n-1} x$$

γ_{ij} - metric on the hypersurface $\partial\Sigma$

n_μ - normal to the hypersurface

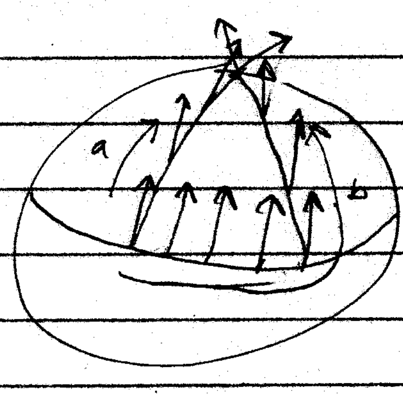
Parallel Transport + Geodesics

covariant derivatives measure the rate of change of tensor fields in comparison to what it would be if it were "parallel transported"

parallel transport - moving a vector (or tensor) along a path but keeping it constant - this is trivial in flat space and implies why the covariant derivative of the metric is always zero

in curved space the path used to parallel transport a vector can affect the result of the parallel transport

2-sphere example



keeping a vector constant implies

$$0 = \frac{d}{d\lambda} T^{\mu_1\mu_2 \cdots \mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l}$$

directional covariant derivative - $\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu$

$$\therefore \left(\frac{D}{d\lambda} T\right)^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_\ell} \equiv \frac{dx^\sigma}{d\lambda} \nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \nu_2 \cdots \nu_\ell} = 0$$

eqn of parallel transport

$$\frac{d}{d\lambda} V^\mu + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} V^\rho = 0$$ - for vectors

for a metric-compatible connection the metric is alway parallel transported w.r.t the connection

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu\nu} = 0$$

$$\therefore \frac{D}{d\lambda}(g_{\mu\nu} V^\mu W^\nu) = \left(\frac{D}{d\lambda} g_{\mu\nu}\right) V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda} V^\mu\right) W^\nu + g_{\mu\nu} V^\mu \left(\frac{D}{d\lambda} W^\nu\right)$$

$$= 0$$

the norm of the vectors + orthogonality is preserved

geodesic - curve-space generalization of a straight line in Euclidean space

- path that parallel-transports its own tangent vector

x^μ - path $\quad \frac{dx^\mu}{d\lambda}$ - tangent vector

$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0$ — parallel transport, tangent vector is const

$\therefore \quad \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$

geodesic eqn

proper time for a path $\quad \tau = \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{\frac{1}{2}} d\lambda \Rightarrow \tau = \int \sqrt{-f}\, d\lambda$

$f = g_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)$

$\delta\tau = \int \delta\sqrt{-f}\, d\lambda = -\frac{1}{2}\int (-f)^{-\frac{1}{2}}\delta f\, d\lambda$

$\lambda \to \tau \quad \frac{dx^\mu}{d\lambda} \to U^\mu$

$\therefore f = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}U^\mu U^\nu = -1$

$\therefore \delta\tau = -\frac{1}{2}\int \delta f\, d\tau \quad$ if $\delta\tau = 0$ - Stationary pts of $\frac{dx^\mu}{d\tau}$

define to find δf $\Rightarrow$ $I = \frac{1}{2}\int f\, d\tau = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\, d\tau$

$x^\mu \to x^\mu + \delta x^\mu$

$g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\delta x^\sigma \leftarrow$ Taylor expansion

$\delta I = \frac{1}{2}\int d\tau \Big[\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}$

$$\frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}d\tau \overset{\text{IBP}}{=} -\frac{1}{2}\int\left[g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}+\frac{dg_{\mu\nu}}{d\tau}\frac{dx^\mu}{d\tau}\right]\delta x^\nu d\tau$$

$$= -\frac{1}{2}\int\left[g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}+\partial_\sigma g_{\mu\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}\right] \times \delta x^\nu d\tau$$

$$\therefore \delta I = -\int\left[g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2}+\frac{1}{2}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] \times \delta x^\nu d\tau$$

$$\delta I = 0 \Rightarrow 0 = g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2}+\frac{1}{2}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$\therefore \frac{d^2x^\rho}{d\tau^2}+\frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$$

or $$\frac{d^2x^\rho}{d\tau^2}+\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$$

Properties

unaccelerated test particles follow geodesics

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}=\frac{q}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau} \leftarrow f=ma$$

$\tau \to \lambda = a\tau + b$ - leaves the eqn invariant

affine parameter - works like proper time for parameterizing the eqn

for timelike paths $\quad U^\lambda\nabla_\lambda U^\mu = 0$ - geodesic eqn for 4-momentum $p^\mu = m U^\mu$

$$p^\lambda \nabla_\lambda p^\mu = 0$$ - geodesic eqn

freely-falling particles keep moving in the direction of their momenta

for null paths $\tau = 0$ + therefore not a good parameter we must choose some other parameter λ

$$p^\mu = \frac{dx^\mu}{d\lambda}$$ - momentum 4-vector for null geodesic

$$E = -p_\mu U^\mu$$ - energy measured by observer U^μ

exponential map - maps tangent space T_p to a region of the manifold containing p using geodesics

unique geodesic

$$\frac{dx^\mu}{d\lambda}(\lambda = 0) = K^\mu$$ where $k \in T_p$, $\lambda(p) = 0$

$$\exp_p : T_p \to M$$ where somewhere on M $\lambda = 1$

$$\exp_p(k) = x^\nu(\lambda = 1)$$ - exponential map

unique solution to geodesic eqn

singularity - place where geodesics appear to end

Use the exponential map to construct locally inertial coords

$$g_{\hat\mu\hat\nu} = g(\hat e_{(\hat\mu)}, \hat e_{(\hat\nu)}) = \eta_{\hat\mu\hat\nu}$$

For a point q close enough to p there is a unique geodesic path from p to q + a unique parameterization λ such that $\lambda(p) = 0$ + $\lambda(q) = 1$

$$k = k^{\hat\mu}\hat e_{(\hat\mu)} \quad . \quad x^{\hat\mu}(q) = k^{\hat\mu}$$

components of $k^{\hat\mu}$ that get mapped to q by $\exp_p$

Riemann normal coords at p

verify:

$$x^{\hat\mu}(\lambda) = \lambda k^{\hat\mu} \Rightarrow \frac{d^2x^\mu}{d\lambda^2} = 0$$

true along any geodesic p in this system

$$\frac{d^2x^{\hat\mu}}{d\lambda^2}(p) = -\Gamma^{\hat\mu}_{\hat\rho\hat\sigma}(p)k^{\hat\rho}k^{\hat\sigma}$$

here. $k^{\hat\rho} = \frac{dx^{\hat\rho}}{d\lambda}(p)$

$$\therefore \Gamma^{\hat\mu}_{\hat\rho\hat\sigma}(p) = 0 \Rightarrow 0 = \nabla_{\hat\sigma} g_{\hat\mu\hat\nu} = \partial_{\hat\sigma} g_{\hat\mu\hat\nu}$$