Expanding Universe - Example of a physical manifold

4D cosmological spacetime

$$ds^2 = -dt^2 + \underbrace{a(t)}_{\text{scale factor}}{}^2 [dx^2 + dy^2 + dz^2]$$

Space is an expanding flat 3D Euclidean space

comoving worldlines - remain at const spatial coords x^i

use $a(t) = t^q$ $\quad 0 < q < 1$

matter-dominated flat universe $q = \frac{2}{3}$
radiation-dominated flat universe $q = \frac{1}{2}$

$a(t) \to 0$ as $t \to 0$ the metric becomes singular
"Big Bang"

i.e. $0 < t < \infty$

for light cones (in 1+1 dimensions)

$$ds^2 = 0 = -dt^2 + t^{2q} dx^2$$

$$\Rightarrow \frac{dx}{dt} = \pm t^{-q}$$

integrate + solve for t $\Rightarrow$; $t = (1-q)^{1/(1-q)} (\pm x - x_0)^{1/(1-q)}$

this gives the plot on p 78 Fig 2.22

Note light cones are tangent to the singularity at $t = 0$

light cones from different pts may not intersect in the past

horizon - outside there are world lines that can not influence what happens at the event (out of causal contact)

Causality

Initial-value problems: given the state of a system at some time, what will be the state at a later time?

causality makes it possible to have definite answers to this question

Future events depend on: initial conditions and physical laws

Since GR modifies spacetime can causality be violated?

- evolve matter fields on a fixed background not spacetime itself
- signals travel on timelike or null trajectories

causal curve - curve that is timelike or null everywhere

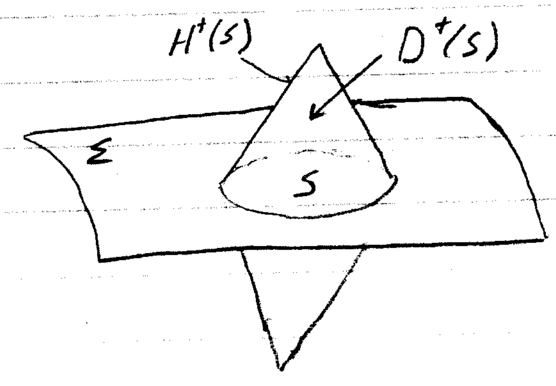
causal future - $J^+(S)$ where $S \subset M$

chronological future - $I^+(S)$ set of pts that can be reached by following a future-directed timelike curve

achronal - if for $S \subset M$ no two pts are connected by a timelike curve

future domain of dependence - $D^+(S)$ set of all pts p such that every past moving (infinitely long) causal curve through p can intersect S

future Cauchy horizon - $H^+(S)$ boundary of $D^+(S)$



Signals can't propogate outside the light cone at any pt p

$\therefore$ every curve must intersect S

$\therefore$ info at S can predict the situation at p

initial data at S can be used to solve for the value at p

Domain of dependence - $D(S) = D^+(S) \cup D^-(S)$

Cauchy Surface - $D(\Sigma)$ is the entire manifold

globally hyperbolic - spacetime has a Cauchy surfac

closed timelike curves - a timelike curve that can intersect w/ it's past because of the curvature of spacetime

Example $\mathbb{R} \times S^1$

$$ds^2 = -\cos\lambda\, dt^2 - \sin\lambda\,[dt\,dx + dx\,dt] + \cos\lambda\, dx^2$$

$$\lambda = \cot^{-1} t \qquad \lambda(t=-\infty) = 0$$
$$\lambda(t=\infty) = \pi$$

as t goes from $-\infty$ to 0 the light cones tilt

at $t > 0$ x becomes timelike and it is possible to travel around S^1 to itself

there is a Cauchy horizon at $t = 0$

Singularities - pts not in the manifold although they can be reached by traveling on a geodesic for a finite distance (curvature becomes infinite)

singularities lead to Cauchy horizons

Tensor Densities

Levi-Civita

$$\tilde{\varepsilon}_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} +1 & \mu_1\mu_2\cdots\mu_n \text{ even permutation} \\ -1 & \mu_1\mu_2\cdots\mu_n \text{ odd permutation} \\ 0 & \text{any reapeated terms} \end{cases}$$

defined not to change under coord transforms

$$\tilde{\varepsilon}_{\mu_1'\mu_2'\cdots\mu_n'} |M| = \tilde{\varepsilon}_{\mu_1\mu_2\cdots\mu_n} M^{\mu_1}{}_{\mu_1'} M^{\mu_2}{}_{\mu_2'} \cdots M^{\mu_n}{}_{\mu_n'}$$

$$\therefore \tilde{\epsilon}_{\mu_1' \mu_2' \cdots \mu_n'} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \tilde{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

since $M^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$

$|M^{-1}| = |M|^{-1}$ det of inverse = inverse of det

this is the definition of a tensor density

weight of tensor density

$$g(x^{\mu'}) = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-2} g(x^{\mu})$$

to convert a density to a tensor multiply by $|g|^{w/2}$

$$\therefore \epsilon_{\mu_1 \mu_2 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n}$$

$$\epsilon^{\mu_1 \mu_2 \cdots \mu_n} = \frac{1}{\sqrt{|g|}} \tilde{\epsilon}^{\mu_1 \mu_2 \cdots \mu_n}$$

$$\epsilon^{\mu_1 \mu_2 \cdots \mu_p \alpha_1 \cdots \alpha_{n-p}} \epsilon_{\mu_1 \mu_2 \cdots \mu_p \beta_1 \cdots \beta_{n-p}} = (-1)^s p!(n-p)!\, \delta^{[\alpha_1}_{\beta_1} \cdots \delta^{\alpha_{n-p}]}_{\beta_{n-p}}$$

Differential Forms

differential p-form - $(0,p)$ antisymmetric tensor

- scalars - 0-forms
- duals - 1-forms
- $\epsilon_{\mu\nu\rho\sigma}$ - 4-form

$\Lambda^p(M)$ - space of p-forms on Manifold M

# of linearly ind p-forms in n-dims
is $n!/(p!(n-p)!)$

in 4D there are

1 lin ind 0-forms
4 lin ind 1-forms
6 lin ind 2-forms
4 lin ind 3-forms
1 lin ind 4-forms

A - p-form B - q-form

wedge product

$A \wedge B$ - $(p+q)$ form

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$$

$$\therefore (A \wedge B)_{\mu\nu} = 2 A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$$

similar to $\times$ product

exterior derivative - differentiate p-form fields
to obtain $(p+1)$ form fields

appropriately normalized antisymmetrized partial derivative

$(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}$

Gradient - exterior derivative of a 0-form

$(d\phi)_\mu = \partial_\mu \phi$

$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta$

(2.38) $\rightarrow W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} = W_\nu \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}}$ symmetric but the exterior deriv is anti-sym by definition

$\therefore$ the exterior deriv is a tensor while the partial isn't

$d(dA) = 0$ ← since partial derivs commute

closed if $dA = 0$ exact if $A = dB$

Hodge duality - maps of p-forms to $(n-p)$ forms

$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p}$

$**A = (-1)^{s + p(n-p)} A$

s - # of minus signs in metric eigenvalues

$*(U \wedge V)_i = \epsilon_i{}^{jk} U_j V_k$ ← same as $\times$-product

Electrodynamics

$$\partial_{[\mu}F_{\nu\lambda]} = d\underset{\text{2-form}}{F} = 0$$

in Minkowski space all closed forms are exact

$\therefore\ F = dA$

A: 1-form vector potential

or

0-form scalar potential

$$d(*F) = *J$$

if $J_\mu = 0$ $\quad F \to *F$

$\quad *F \to -F$

Integration

$$\text{in } \mathbb{R}^n \quad d^n x' = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| d^n x$$

element of Jacobian

$\int_\Sigma : \omega \to \mathbb{R}$ the integral maps an n-form ω to real nos

$$\int \underbrace{\overset{\text{1-form}}{\omega(x)}\, dx}_{\text{1-form}}$$

diff form

In more than 1-D

Volume elements are n-forms

$$d^n x = dx^0 \wedge \cdots \wedge dx^{n-1}$$

← tensor density

~~→ coord dependent~~

$$\Rightarrow dx^0 \wedge \cdots \wedge dx^{n-1} = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

$$\tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| \tilde{\epsilon}_{\mu_1' \cdots \mu_n'} dx^{\mu_1'} \wedge \cdots \wedge dx^{\mu_n'}$$

$$\sqrt{|g'|}\, dx^{0'} \wedge \cdots \wedge dx^{(n-1)'} = \sqrt{|g|}\, dx^0 \wedge \cdots \wedge dx^{n-1}$$

↑ invarient volume element

or $\sqrt{|g|}\, d^n x \equiv \sqrt{|g|}\, dx^0 \wedge \cdots \wedge dx^{n-1}$

$$\Rightarrow I = \int \phi(x) \sqrt{|g|}\, d^n x$$