Vectors (again)

function ↘ $f : M \to \mathbb{R}$ is C^∞ ↙ manifold

if $f : M \to \mathbb{R}$ is C^∞

a tangent space at pt p in the Manifold defines the directional derivatives

$$f \to \frac{df}{d\lambda} \text{ at } p \text{ parameterized by } \lambda$$

if we paramaterize by λ & η

coordinates → $x^\mu(\lambda)$ & $x^\mu(\eta) \longrightarrow a\frac{df}{d\lambda} + b\frac{df}{d\eta}$

the chain rule applies

$$\left(a\frac{d}{d\lambda} + b\frac{d}{d\eta}\right)(fg) = af\frac{dg}{d\lambda} + ag\frac{df}{d\lambda} + bf\frac{dg}{d\eta} + bg\frac{df}{d\eta}$$

$$= \left(a\frac{df}{d\lambda} + b\frac{df}{d\eta}\right)g + \left(a\frac{dg}{d\lambda} + b\frac{dg}{d\eta}\right)f$$

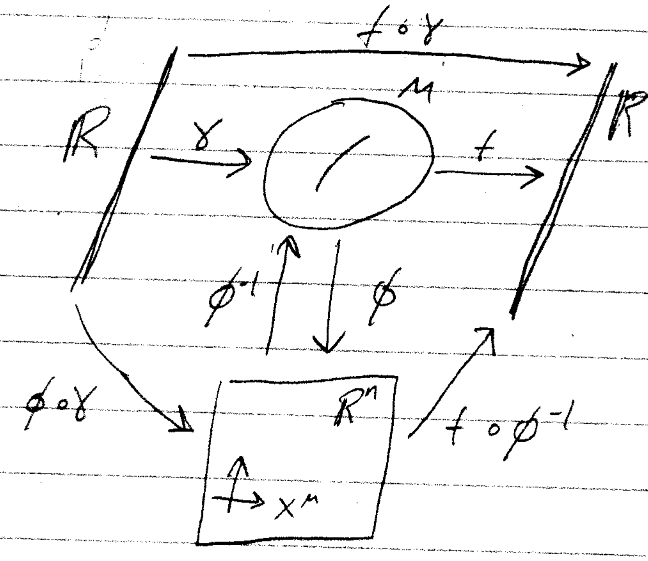
x^μ - coordinate

$\{\partial_\mu\}$ - partial derivatives which form the basis of T_p

$\phi : M \to \mathbb{R}^n$ - coordinate chart

$\gamma : \mathbb{R} \to M$ - curve - paramaterized by λ

$f : M \to \mathbb{R}$ - function



$$\frac{d}{d\lambda} f = \frac{d}{d\lambda}(f \circ \gamma) = \frac{d}{d\lambda}\left[(f \circ \phi^{-1}) \circ (\phi \circ \gamma)\right]$$

$$= \frac{d(\phi \circ \gamma)^\mu}{d\lambda} \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu} \quad \leftarrow \text{chain rule}$$

$$= \frac{dx^\mu}{d\lambda} \partial_\mu f$$

$$\therefore \quad \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu \qquad \partial_\mu = \hat{e}_{(\mu)}$$

coordinate basis for T_p

Note: coord basis vectors are not always normalized to unity or orthogonal. On a curved manifold a coord basis is never orthonormal throughout the neighborhood of a pt where curvature doesn't vanish transformation law

$$\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$

$$V^{\mu}\partial_{\mu} = V^{\mu'}\partial_{\mu'} = V^{\mu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu}$$

$$\therefore V^{\mu} = V^{\mu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}$$

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$

more general
than Lorentz
Transforms
they can be non-linear

Commutator $[X,Y]$ - $X + Y$ are vector fields

linear opperator $\rightarrow$ $[X,Y](f) \equiv X(Y(f)) - Y(X(f))$

vector fields map smooth functions to smooth functions all over the manifold by taking a derivative at every point

$$[X,Y]^{\mu} = X^{\lambda}\partial_{\lambda}Y^{\mu} - Y^{\lambda}\partial_{\lambda}X^{\mu}$$

Note: Partials commute ∂_{μ}

Tensors (again)

$\omega : T_p \to \mathbb{R}$ - maps tangent space to a set of real no's

df - gradient of f, is a one-form

$d/d\lambda$ - vector

$$\therefore \quad df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} \text{ - scalar}$$

The gradients of coords dx^μ form a natural basis for T_p^* bc $\hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) = \delta^\mu_\nu$

$$\therefore dx^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

$$\omega = \omega_\mu dx^\mu$$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

$$\Rightarrow \omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T(dx^{\mu_1}, \cdots dx^{\mu_k}, \partial_{\nu_1}, \cdots \partial_{\nu_l})$$

$$\therefore T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$$

Example

(0,2) Symmetric tensor $S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}$

$$S = S_{\mu\nu}(dx^\mu \otimes dx^\nu)$$

$$\doteq (dx)^2 + x^2(dy)^2$$

choose $x' = \frac{2x}{y}$ $y' = \frac{y}{2}$ where $x > 0 + y > 0$

$$x = \tfrac{1}{2}x'y \quad y = 2y' \;,\; x = x'y'$$

$$dx = y'dx' + x'dy'$$
$$dy = 2dy'$$

$$\Rightarrow S = (y'dx' + x'dy')^2 + (x'y')^2(2dy')^2$$

$$\therefore S_{\mu'\nu'} = \begin{pmatrix} y'^2 & x'y' \\ x'y' & x'^2 + 4(x'y')^2 \end{pmatrix}$$

Most tensors are the same under contraction, symmetrization etc. . .

exceptions are the metric, partial derivatives and Levi-Civita for partial derivatives

$$\partial_{\mu'} W_{\nu'} = \frac{\partial}{\partial x^{\mu'}} W_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu}}{\partial x^{\nu'}} W_{\nu} \right)$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \left(\frac{\partial}{\partial x^{\mu}} W_{\nu} \right)$$

$$+ W_{\nu} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}}$$

depends on the partial deriv of the transformation matrix i. not a tensor doesn't transform as a tensor

The Metric

symmetric (0,2) tensor usually nondegenerate

$$g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu} = \delta^{\mu}_{\sigma}$$

raises + lowers indices

other uses

1) supplies a notion of "past" and "future"
2) allows the computation of path length + proper time
3) determines the "shortest distance" between pts
4) replaces the Newtonian gravitational field
5) provides a notion of locally inertial (non rotating) frames
6) determines causality
7) replaces the dot-product

path length in flat space $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

path length in general space $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

in cartesian coords

$$ds^2 = dx^2 + dy^2 + dz^2$$

in spherical coords

$x = r\sin\theta\cos\phi$
~~$y = r\sin\theta\sin\phi$~~
$z = r\cos\theta$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$\Rightarrow g_{\mu\nu} V^\mu W^\nu = g(V,W) = ds^2(V,W)$$

signature - # of positive + negative eigenvalues
$(-,+,+,+)$

GR + SR

Euclidean or Riemannian - all positive
Lorentzian or pseudo-Riemannian - one minus
indefinite - some +1, some -1

Locally inertial coords - $g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}$

Local Lorentz frame $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$

$$\partial_{\hat{\rho}} \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) \neq 0$$

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu}$$

expand in Taylor series

$$x^{\mu} = \left(\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\right)_p x^{\hat{\mu}} + \frac{1}{2}\left(\frac{\partial^2 x^{\mu}}{\partial x^{\hat{\mu}_1}\partial x^{\hat{\mu}_2}}\right)_p x^{\hat{\mu}_1} x^{\hat{\mu}_2}$$

$$+ \frac{1}{6}\left(\frac{\partial^3 x^{\mu}}{\partial x^{\hat{\mu}_1}\partial x^{\hat{\mu}_2}\partial x^{\hat{\mu}_3}}\right)_p x^{\hat{\mu}_1} x^{\hat{\mu}_2} x^{\hat{\mu}_3} + \cdots$$

assume $x^{\mu}(p) = x^{\hat{\mu}}(p) = 0$ for simplicity

$$\Rightarrow (\hat{g})_p + (\hat{\partial}\hat{g})_p \hat{x} + (\hat{\partial}\hat{\partial}\hat{g})_p \hat{x}\hat{x} = \left(\frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} g\right)_p$$

$$+ \left(\frac{\partial x}{\partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x}\partial \hat{x}} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} \hat{\partial} g\right)_p \hat{x}$$

$$+ \left(\frac{\partial x}{\partial \hat{x}} \frac{\partial^3 x}{\partial \hat{x}\partial \hat{x}\partial \hat{x}} g + \frac{\partial^2 x}{\partial \hat{x}\partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x}\partial \hat{x}} g\right.$$

$$\left. + \frac{\partial x}{\partial \hat{x}} \frac{\partial x^2}{\partial \hat{x}\partial \hat{x}} \hat{\partial} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} \hat{\partial}\hat{\partial} g\right)_p \hat{x}\hat{x}$$

Set terms of equal $\hat{x}$ equal to each other

$\therefore$ $g_{\hat{\mu}\hat{\nu}}(p)$ are determined by $\left(\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\right)_p$ — 16 total terms $\therefore$ possible

set $\hat{\partial}\hat{g} = 0$ — 40 total terms on RHS
$\therefore$ possible

Example:

What does an observer measure as ordinary 3-velocity?

transform to locally inertial coords
set the observer in the rest frame

$$U^{\hat{\mu}} = (1, 0, 0, 0) \text{ - observer}$$

$$V^{\hat{\mu}} = (\gamma, v\gamma, 0, 0) \text{ - rocket}$$

$$\gamma = 1/\sqrt{1-v^2} \Rightarrow v = \sqrt{1-\gamma^{-2}}$$

$$\gamma = -g_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} V^{\hat{\nu}} = -U_{\hat{\mu}} V^{\hat{\mu}}$$

$$\therefore v = \sqrt{1-(U_{\hat{\mu}} V^{\hat{\mu}})^{-2}}$$

convert back to curved space

$$\Rightarrow v = \sqrt{1-(U_{\mu} V^{\mu})^{-2}}$$

contraction is the same in either coords