Manifolds

Weak Equivalence Principle (WEP) - inertial mass and gravitational mass are equivalent

$$\vec{F} = m_i \vec{a} \qquad \vec{F}_g = -m_g \nabla \Phi$$

$$m_i = m_g \quad \therefore \quad \vec{a} = -\nabla \Phi$$

This suggests that there are a preferred class of trajectories through spacetime; inertial (freely-falling) trajectories

subject only to gravity

every particle has the same gravitational charge

$\frac{m_g}{m_i} = 1$ - this is not true for other forces

WEP - there is no way to distigish between being in a gravitational field or an accelerating reference fram in a 'local space

Note in an accelerating frame objects move in a straight line in a gravitational field they move towards the middle of the field

Einstein Equivalence Principle (EEP) - "In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field using local experiments"

for nongravitational laws of physics

Hydrogen Mass < Proton + electron mass

∴ the binding energy has negative mass

∴ the EM binding energy affects gravity

Strong Equivalence Principle (SEP) - is a version of EEP defined for all the laws of physics not just gravity

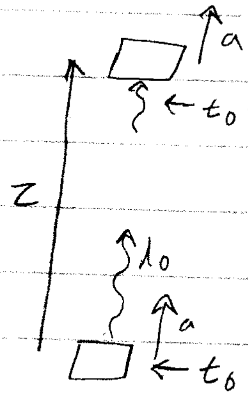
unaccelerated = freely falling

locally inertial frames - follow the motion of individual freely falling particles in small enough regions of spacetime

differential manifold - mathematical structure used to describe curvature

construct a curved spacetime and see if it is equivalent to gravity

Gravitational redshift

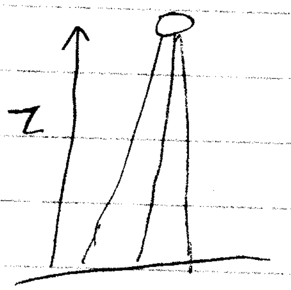


$\Delta v = a\,\Delta t = \dfrac{az}{c}$

$\dfrac{\Delta v}{c} << 1$

$\dfrac{\Delta\lambda}{\lambda_0} = \dfrac{\Delta v}{c} = \dfrac{az}{c^2}$ Doppler effect to first order

according to EEP this should also work in a uniform gravitational field



$\downarrow a_g$ $\quad \therefore \dfrac{\Delta\lambda}{\lambda_0} = \dfrac{a_g z}{c^2}$

$dt = \frac{1}{c} dz$

$a_g = \nabla\Phi \Rightarrow \dfrac{\Delta\lambda}{\lambda_0} = \dfrac{1}{c}\underbrace{\int \nabla\Phi\, dt}_{\Delta v} = \dfrac{1}{c^2}\int \partial_z \Phi\, dz = \dfrac{\Delta\Phi}{c^2}$

if an observer at the top measures λ_1 + an observer at the base measures λ_0 where $\lambda_1 = \lambda_0 + \Delta\lambda$

$\Delta t_0 = \lambda_0/c \quad \Delta t_1 = \lambda_1/c \quad \therefore \Delta t_1 > \Delta t_0$

the time it takes a wavelength to pass is increased

Manifold - mathematical representation of an n-dimensional space that may be curved but looks like $\mathbb{R}^n$ Euclidean space in local regions

We can analyze functions on a space by mapping them to functions in Euclidean space

Examples:

$\mathbb{R}^n$ - look like $\mathbb{R}^n$ everywhere; $\mathbb{R}^1$ line, $\mathbb{R}^2$ plane..

S^n - n-sphere - locus of all pts a fixed distance from the origin in $\mathbb{R}^{n+1}$

S^1 - circle, S^2 - sphere, ect..

S^0 - disconnected zero-dim manifold

T^n - n-torus - n-dim cube w/ opposite sides touching

Riemann surface of genus g - T^2 w/ g holes instead of one

$\therefore$ S^2 = Riemann surface w $g = 0$

Continuous transformations in $\mathbb{R}^n$ - Lie groups are manifolds w/ group structure

$\therefore$ $SO(2) = S^1$

Direct Product of 2 manifolds is a manifold. If M has dim n + M' has dim n', $M \times M'$ has dimension $n+n'$

Manifolds must look locally like $\mathbb{R}^n$ everywhere

We will only deal w/ manifolds that have continuous derivatives (smooth)

Maps:

If M + N are sets $\phi : M \to N$ is a map it assigns an element of N to each element of M or function of elements in the set

Given $\phi : A \to B$ $\psi : B \to C$

$\psi \circ \phi : A \to C$ - composition

$$\begin{tikzcd} A \arrow[rr, "\psi \circ \phi"] \arrow[dr, "\phi"'] & & C \\ & B \arrow[ur, "\psi"'] & \end{tikzcd}$$

$(\psi \circ \phi)(a) = \psi(\phi(a))$

$a \in A$, $\phi(a) \in B$ $\therefore$ $\psi(\phi(a)) \in C$

one-to-one - each element in N has no more than one element of M mapped to it

onto - each element in N has at least one element of M mapped into it

$\phi(x) = e^x$ - one-to-one but not onto

$\phi(x) = x^3 - x$ - onto but not one-to-one

$\phi(x) = x^3$ - one-to-one + onto

$\phi(x) = x^2$ - neither

domain of the map - M
image of the map - where the domain gets mapped to
preimage - set of images of M that get mapped to $U \subset N$ (subset of N)
invertible - map that is one-to-one and onto
inverse map - $\phi^{-1} : N \to M$ i.e. $(\phi^{-1} \circ \phi)(a) = a$

Continuity

$\phi : \mathbb{R}^m \to \mathbb{R}^n$

Assume a set of n-functions w/ m variables

$$y^1 = \phi^1(x^1, x^2, \dots x^m)$$
$$y^2 = \phi^2(x^1, x^2, \dots x^m)$$
$$\vdots$$
$$y^n = \phi^n(x^1, x^2 \dots x^m)$$

C^p - p^{th} derivative of the function exists and is continuous

C^0 - continuous but not necessarily differentiable

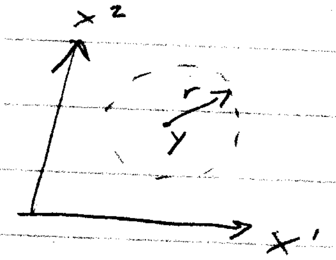
smooth → C^∞ - continuous and infinity differentiable diffeomorphic - C^∞ map $\phi : M \to N$ with C^∞ inverse $\phi^{-1} : N \to M$ exists

here ϕ is a diffeomorphism

if 2 spaces are "the same" as manifolds they are diffeomorphic

open ball - set of all pts x in $\mathbb{R}^n$ such that $|x - y| < r$ where $y \in \mathbb{R}^n$ $r \in \mathbb{R}$

$$|x - y| = \left(\sum_i (x^i - y^i)^2\right)^{\frac{1}{2}}$$



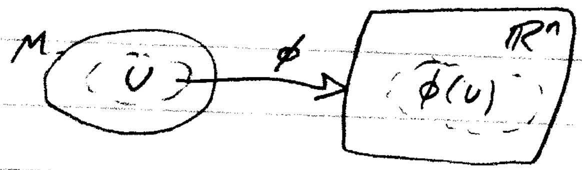
open set - is a set constructed from an arbitrary union of open balls

$V \subset \mathbb{R}^n$ $\quad y \in V$

i.e. an open set is the interior of an $(n-1)$ dim closed surface

chart or coordinate system - $\underset{\text{subset of } M}{U \subset M}$ + $\underset{\text{one-to-one map}}{\phi : U \to \mathbb{R}^n}$

V is an open set in M.



atlas - indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$

1. The union of the U_α is equal to M
   U_α cover M
2. The charts are "smoothly sewn together"
   see fig 2.14

C^∞ n-dim manifold is a set M along w/
a maximal atlas and contains every possible chart