Classical Field Theory (Not Quantum) ③

dynamic tensor field

$SR \to GR : \eta_{\mu\nu} \to g_{\mu\nu}(\vec{x})$

from the "principle of least action"

$$\underset{\text{action}}{S} = \int dt \, \underset{\text{Lagrangian}}{L(q,\dot{q})}$$

$$L = \underset{\substack{\text{kinetic}\\\text{energy}}}{T} - \underset{\substack{\text{potential}\\\text{energy}}}{V}$$

$$\delta S = 0 = \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)$$

Euler-Lagrange Eqns

$\therefore$ if $L = \frac{1}{2}\dot{q}^2 - V(q)$

$$\ddot{q} = -\frac{dV}{dq}$$

in field theory

$$q(t) \to \cancel{\Phi}(x^\mu)$$

for E+M $\Phi^{\sigma} = \{A_0, A_1, A_2, A_3\}$

$$L = \int d^3x \, \mathcal{L}(\Phi^{\sigma}, \partial_\mu \Phi^{\sigma})$$

Lagrange density

$$S = \int dt \, L = \int d^4x \, \mathcal{L}(\Phi^{\sigma}, \partial_\mu \Phi^{\sigma})$$

natural units – $c = 1$ $\hbar = k = 1 = \frac{h}{2\pi}$

[energy] = [mass] = [(length)$^{-1}$] = [(time)$^{-1}$]

$[S] = [ET] \Rightarrow$ dimensionless $= M^{0}$ zero power

mass

$[d^4x] = M^{-4}$

$[\mathcal{L}] = M^4$

Variations in the field

$$\Phi^{\sigma} \to \Phi^{\sigma} + \delta\Phi^{\sigma}$$

$$\partial_\mu \Phi^{\sigma} \to \partial_\mu \Phi^{\sigma} + \delta(\partial_\mu \Phi^{\sigma}) = \partial_\mu \Phi^{\sigma} + \partial_\mu(\delta \Phi^{\sigma})$$

$$= \partial_\mu(\Phi^{\sigma} + \delta\Phi^{\sigma})$$

$$\mathcal{L}(\Phi^{\sigma}, \partial_\mu \Phi^{\sigma}) \to \mathcal{L}(\Phi^{\sigma} + \delta\Phi^{\sigma}, \partial_\mu \Phi^{\sigma} + \partial_\mu \delta\Phi^{\sigma})$$

$$= \mathcal{L}(\Phi^{\sigma}, \partial_\mu \Phi^{\sigma}) + \frac{\partial \mathcal{L}}{\partial \Phi^{\sigma}} \delta \Phi^{\sigma} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^{\sigma})} \partial_\mu (\delta \Phi^{\sigma})$$

$$S \to S + \delta S$$

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^{\sigma}} \delta \Phi^{\sigma} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^{\sigma})} \partial_\mu (\delta \Phi^{\sigma}) \right]$$

$$IBP \to \int d^4x \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^{\sigma})} \partial_\mu (\delta \Phi^{\sigma}) = - \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^{\sigma})} \right) \delta \Phi^{\sigma}$$

$$+ \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^{\sigma})} \delta \Phi^{\sigma} \right)$$

total derivative
~~use stoke's theorem~~

since some terms vanish at infinity, we continue IBP until

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^{\sigma}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^{\sigma})} \right) \right] \delta \Phi^{\sigma}$$

$$\delta S = \int d^4x \frac{\delta S}{\delta \Phi^{\sigma}} \delta \Phi^{\sigma}$$

Euler-Lagrange for field theory in flat space

$$\therefore \frac{\delta S}{\delta \Phi^{\sigma}} = \frac{\partial \mathcal{L}}{\partial \Phi^{\sigma}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^{\sigma})} \right) = 0$$

Real Scalar field: (simpiliest example)

$\phi(x^\mu)$: (space time) $\to \mathbb{R}$

upon quantization excitations of the field are observable as particles

Scalar fields $\to$ spinless particles

Vector + Tensor fields $\to$ higher spin particles

complex field $\to$ particle + antiparticle

For a real scalar field

KE: $\frac{1}{2}\dot{\phi}^2$ Not Lorentz invarient

Gradient Energy: $\frac{1}{2}(\nabla\phi)^2$

potential energy: $V(\phi)$ $\leftarrow$ Lorentz invarient

$$-\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2$$

Lorentz invarient

$$\therefore \mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

M^1 $\qquad$ M^4

i.e. $\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{dV}{d\phi}$, $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\eta^{\mu\nu}\partial_\nu \phi$

$$\Rightarrow \frac{\delta S}{\delta \Phi^\sigma} = \frac{\partial \mathcal{L}}{\partial \Phi^\sigma} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\sigma)}\right) = \Box\phi - \frac{dV}{d\phi} = 0$$

$$\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$$

$$\Rightarrow \ddot{\phi} - \nabla^2\phi + \frac{dV}{d\phi} = 0$$ – in flat space

if $V(\phi) = \frac{1}{2}m^2\phi^2$

$$\Rightarrow \Box\phi - m^2\phi = 0$$ — Klein-Gordon eqn

EM example:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Note: if: $A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x)$

$$\therefore F_{\mu\nu} \rightarrow F_{\mu\nu} + \partial_\mu\partial_\nu\lambda - \partial_\nu\partial_\mu\lambda = F_{\mu\nu}$$

$$\partial_{[\mu}F_{\nu\sigma]} = \partial_{[\mu}\partial_\nu A_{\sigma]} - \partial_{[\mu}\partial_\sigma A_{\nu]} = 0$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \delta_\mu^\nu J^\mu = J^\nu$$

$$F_{\mu\nu} F^{\mu\nu} = \eta^{\alpha\rho}\eta^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma}$$

$$\frac{\partial (F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = \eta^{\alpha\rho}\eta^{\beta\sigma}\left[\left(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\right)F_{\rho\sigma} + F_{\alpha\beta}\left(\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}\right)\right]$$

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu$$

$$\therefore \frac{\partial (F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = \eta^{\alpha\rho}\eta^{\beta\sigma}\left[(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu)F_{\rho\sigma} + (\delta_\rho^\mu \delta_\sigma^\nu - \delta_\sigma^\mu \delta_\rho^\nu) F_{\alpha\beta}\right]$$

$$= (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}) F_{\rho\sigma}$$

$$+ (\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}) F_{\alpha\beta}$$

$$= F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu}$$

$$= 4F^{\mu\nu}$$

$$\therefore \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\partial_\mu F^{\nu\mu} = J^\nu$$

Energy-momentum tensor for a scalar theory

$$T^{\mu\nu}_{scalar} = \eta^{\mu\lambda}\eta^{\nu\sigma}\,\partial_\lambda\phi\,\partial_\sigma\phi - \eta^{\mu\nu}\left[\frac{1}{2}\eta^{\lambda\sigma}\partial_\lambda\phi\,\partial_\sigma\phi + V(\phi)\right]$$

E-M tensor for EM field

$$T^{\mu\nu}_{EM} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$$

$\partial_\mu T^{\mu\nu} = 0$ — energy-momentum conserved