Tensors ②

tensor - generalization of vectors and dual vectors
maps multiple vectors and/or dual vectors to $\mathbb{R}$

$$T : T^*_{p_1} \times \cdots \times T^*_{p_k} \times T_{p_1} \times \cdots T_{p_l} \to \mathbb{R}$$

linear in all arguments

$$T(a\omega + b\eta, cV + dW) = ac\,T(\omega, V) + ad\,T(\omega, W) + bc\,T(\eta, V) + bd\,T(\eta, W)$$

scalar type (0,0) tensor
vector type (1,0) tensor
dual vector type (0,1) tensor

tensor product $\otimes$: if T is a (k,l) and S is (m,n)
$T \otimes S$ is $(k+m, l+n)$

$$T \otimes S \neq S \otimes T$$

basis tensors : $\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)}$

$$\therefore T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)}$$

$$T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \omega^{(1)}_{\mu_1} \cdots \omega^{(k)}_{\mu_k} V^{(1)\nu_1} \cdots V^{(l)}$$

$$T^{\mu_1'\cdots\mu_k'}{}_{\nu_1'\cdots\nu_l'} = \Lambda^{\mu_1'}{}_{\mu_1}\cdots\Lambda^{\mu_k'}{}_{\mu_k}\Lambda^{\nu_1}{}_{\nu_1'}\cdots\Lambda^{\nu_l}{}_{\nu_l'}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$$

upper indices transform like vectors
lower indices transform like dual vectors

$T^\mu{}_\nu : V^\nu \rightarrow T^\mu{}_\nu V^\nu$ map from Vector to Tensor and Vector

inner (dot or scalar) product -

in flat space $\eta(V,W) = \eta_{\mu\nu}V^\mu W^\nu = V\cdot W$

orthogonal - inner product is zero

norm - inner product of the vector w/ itself

$$\eta_{\mu\nu}V^\mu V^\nu \begin{cases} < 0 & \text{timelike} \\ = 0 & \text{~~lightlike~~ or null} \\ > 0 & \text{spacelike} \end{cases}$$

Kronecker delta - identity map

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^\mu_\rho$$

Levi-Civita Symbol

Tensor density not tensor $\rightarrow \tilde{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{even permutations} \\ -1 & \text{odd permutations} \\ 0 & \text{repeated indices} \end{cases}$

Tensors in Flat Space:

metric, inverse metric, Kronecker delta, and Levi-Civita symbol are the same in any inertial coordinate system in flat spacetime

EM field strength tensor -

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}$$

Manipulating Tensors

contraction: $S^{\mu\rho}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu} \overset{?}{=} T^{\mu\rho\nu}{}_{\sigma\nu}$

raising + lowering indices:

$$T^{\alpha\beta\mu}{}_{\delta} = \eta^{\mu\gamma} T^{\alpha\beta}{}_{\gamma\delta}$$

$$T_{\mu}{}^{\beta}{}_{\gamma\delta} = \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\delta}$$

$$V_\mu = \eta_{\mu\nu} V^\nu$$

$$\omega^\mu = \eta^{\mu\nu} \omega_\nu$$

$$A^\lambda B_\lambda = \eta^{\lambda\rho} A_\rho \eta_{\lambda\sigma} B^\sigma = \delta^\rho_\sigma A_\rho B^\sigma = A_\sigma B^\sigma$$

Symmetric : $S_{\mu\nu\rho} = S_{\nu\mu\rho}$

Antisymmetric : $A_{\mu\nu\rho} = -A_{\rho\nu\mu}$

Symmetrize tensor :

$$\underset{\substack{\uparrow \\ \text{symmetric}}}{T_{(\mu_1\mu_2\cdots\mu_n)\rho}}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{sum of permutations of indices } \mu_1\cdots\mu_n\right)$$

Antisymmetrize tensor :

$$\underset{\substack{\uparrow \\ \text{antisymmetric}}}{T_{[\mu_1\mu_2\cdots\mu_n]\rho}}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{alternating } (\pm) \text{ sum over permutations of indices } \mu_1\cdots\mu_n\right)$$

$$T_{(\mu|\nu\sigma|\rho)} = \frac{1}{2}\left(T_{\mu\nu\sigma\rho} + T_{\rho\nu\sigma\mu}\right)$$

Not included in sum

$$T_{\mu\nu\rho\sigma} = T_{(\mu\nu)\rho\sigma} + T_{[\mu\nu]\rho\sigma}$$

$$T_{\mu\nu\rho\sigma} \neq T_{(\mu\nu\rho)\sigma} + T_{[\mu\nu\rho]\sigma}$$

trace : $X = X^{\lambda}{}_{\lambda}$ ← sum of diagonal components

partial derivatives : ∂_μ linear in flat space but not in general

Maxwell's Eqns

$$\nabla \times \vec{B} - \partial_t \vec{E} = \vec{J} \qquad \nabla \times \vec{E} + \partial_t \vec{B} = 0$$

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \cdot \vec{B} = 0$$

or

$$\tilde{\epsilon}^{ijk} \partial_j B_k - \partial_0 E^i = J^i \qquad \tilde{\epsilon}^{ijk} \partial_j E_k + \partial_0 B^i = 0$$

$$\partial_i E^i = J^0 \qquad \partial_i B^i = 0$$

$$F^{0i} = E^i$$
$$F^{ij} = \tilde{\epsilon}^{ijk} B_k$$

$\therefore$ Maxwell's Eqns become

$$\left.\begin{array}{l} \partial_j F^{ij} - \partial_0 F^{0i} = J^i \\ \partial_i F^{0i} = J^0 \end{array}\right\} \partial_\mu F^{\nu\mu} = J^\nu$$

$$\partial_{[\mu} F_{\nu\lambda]} = 0 = \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}$$

Energy and Momentum

single particle worldline - map $\mathbb{R} \to M$

manifold representing spacetime

curve $x^\mu(\lambda)$

tangent to the curve $\frac{dx^\mu}{d\lambda}$

we are interested in the norm of the tangent vector
if the tangent vector is timelike/null/spacelike at
the parameter λ, we say the path is timelike/null
/spacelike at the point

four-velocity: $U^\mu = \frac{dx^\mu}{d\tau}$ $\quad d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$

$$\therefore \eta_{\mu\nu} U^\mu U^\nu = -1$$

Momentum four-vector: $p^\mu = \underset{\substack{\uparrow \\ \text{mass}}}{m} U^\mu$

$E = p^0$

$\therefore E = mc^2$ in the particle's rest frame

$p^\mu = (\gamma m, v\gamma m, 0, 0)$ where $c = 1$
$\gamma = 1/\sqrt{1-v^2}$

$\left.\begin{array}{l} p^0 = m + \frac{1}{2}mv^2 \\ p^1 = mv \end{array}\right\}$ for $v \ll c$

$p_\mu p^\mu = -m^2$

$E = \sqrt{m^2 + p^2}$

in Newtonian Physics $\vec{f} = m\vec{a} = \frac{d\vec{p}}{dt}$

i.e. $f^\mu = m \frac{d^2 x^\mu(\tau)}{d\tau^2} = \frac{d}{d\tau} p^\mu(\tau)$

$\vec{f} = q(\vec{E} + \vec{V} \times \vec{B}) \Rightarrow -q U^\lambda F_\lambda{}^\mu$

Energy Momentum tensor $T^{\mu\nu}$ (considering a fluid system)

T^{00} - flux of p^0 in the x^0 direction
rest-frame energy density ρ

$T^{0i} = T^{i0}$ - momentum density

T^{ij} - momentum flux or stress

T^{ii} - pressure - p_i

Number-flux four-vector - $N^\mu = n U^\mu$

n - number of particles of dust

$\rho = mn$ - rest frame energy density

$T^{\mu\nu}_{dust} = p \otimes N = p^\mu N^\nu = mn U^\mu U^\nu = \rho U^\mu U^\nu$

Notice zero pressure in every direction

perfect fluid - can be completely specified by the diagonal componets of $T^{\mu\nu}$

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \text{ - at rest}$$

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu} \text{ - in general}$$

dust - $p = 0$

isotropic gas - $p = \frac{1}{3}\rho$

vacuum energy - $T^{\mu\nu} = -\rho_{vac}\,\eta^{\mu\nu}$

$\partial_\mu T^{\mu\nu} = 0$ - energy + momentum are conserved

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\rho + p)U^\mu U^\nu + (\rho + p)(U^\nu \partial_\mu U^\mu + U^\mu \partial_\mu U^\nu)$$
$$+ \partial^\nu p$$

$$U_\nu \partial_\mu U^\nu = \frac{1}{2}\partial_\mu(U_\nu U^\nu) = 0 \quad + \; U_\nu U^\nu = -1$$

$$\therefore \; U_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu(\rho U^\mu) - p\,\partial_\mu U^\mu$$

Explaination on next page

$$U_\nu \partial_\mu T^{\mu\nu} = \partial_\mu(\rho+p)\,U^\mu \underbrace{U_\nu U^\nu}_{=-1} + (\rho+p)(\underbrace{U_\nu U^\nu}_{=-1}\partial_\mu U^\mu)$$

$$+ (\rho+p)(U^\mu \cancel{U_\nu \partial_\mu U^\nu}^{\,0}) + U_\nu \partial^\nu p$$

$$= -\partial_\mu\{(\rho+p)\,U^\mu\} + \underbrace{U^\mu \partial_\mu p}_{=\partial_\mu(pU^\mu) - p\partial_\mu U^\mu}$$

$$= -\partial_\mu(\rho U^\mu) - p\,\partial_\mu U^\mu$$

$U^\mu = (1, v^i)$ $\quad |v^i| << 1 \quad p << \rho$ - Non relativistic limit

$$U_\nu \partial_\mu T^{\mu\nu} \rightarrow \partial_t \rho + \nabla\cdot(\rho v) = 0$$

Projection tensor - $P^\sigma{}_\nu = \delta^\sigma_\nu + U^\sigma U_\nu$

$$\left.\begin{array}{l} P^\sigma{}_\nu V_\parallel^\nu = 0 \\ P^\sigma{}_\nu W_\perp^\nu = W_\perp^\sigma \end{array}\right\} \text{ check} \quad \begin{array}{l} V_\parallel^\mu \parallel U^\mu \\ W_\perp^\mu \perp U^\mu \end{array}$$

$$P^\sigma{}_\nu \partial_\mu T^{\mu\nu} = \cancel{(\delta^\sigma_\nu + U^\sigma U_\nu)\,\partial_\mu(\rho+p)U^\mu U^\nu}^{\,0}$$
$$+ (\delta^\sigma{}_\nu + U^\sigma U_\nu)(\rho+p)(U^\nu \partial_\mu U^\mu)$$
$$+ (\delta^\sigma{}_\nu + U^\sigma U_\nu)(\rho+p)(U^\mu \partial_\mu U^\nu)$$
$$+ (\delta^\sigma{}_\nu + U^\sigma U_\nu)\,\partial^\nu p$$

$$P^\sigma_{\ \nu} \partial_\mu T^{\mu\nu} = (\rho+p)\cancel{(U^\sigma \partial_\mu U^\mu)} \cancel{- (\rho+p)}\overset{0}{(U^\sigma \partial_\mu U^\mu)}$$

$$+ (\rho+p)(U^\mu \partial_\mu U^\sigma) + (\rho+p)(\cancel{U^\sigma U^\mu \overset{0}{U_\nu \partial_\mu U^\nu}})$$

$$+ \partial^\sigma p + U^\sigma U_\nu \partial^\nu p$$

$$= \rho U^\mu \partial_\mu U^\sigma + p U^\mu \partial_\mu U^\sigma$$

$$+ U^\sigma U_\mu \partial^\mu p + \partial^\sigma p$$

$$= \rho U^\mu \partial_\mu U^\sigma + U^\mu \partial_\mu (p U^\sigma) + \partial^\sigma p$$

$$= (\rho + p) U^\mu \partial_\mu U^\sigma + U^\sigma U^\mu \partial_\mu p + \partial^\sigma p$$

in the non relativistic limit

$$0 = \rho[\partial_t v + (v \cdot \nabla) v] + \nabla p$$

$$+ v(\partial_t p + v \cdot \nabla p)$$

since v is small and $p \ll \rho$

$$\rho[\partial_t v + (v \cdot \nabla) v] \approx -\nabla p$$