Cosmology

Copernican principle - the universe is the same everywhere (at large scales) we are not the center

Cosmic Microwave Background (CMB) - evidence of this

Isotropy - Manifold is the same in every direction
invarient under rotations

Homogeneity - the metric is the same throughout the manifold
invarient under translations

Homogeneity + Isotropy - are not necessarily related
imply that space is maximally symmetric
" max number of killing vectors

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \text{ - for max sym manifold}$$

$$\text{if } R = 0 \Rightarrow ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$\text{if } R > 0 \text{ de Sitter space; } ds_5^2 = -du^2 + dx^2 + dy^2 + dz^2 + dw$$

$$\text{hyperboloid w/ } -u^2 + x^2 + y^2 + z^2 + w^2 = \alpha^2$$

$$u = \alpha \sinh(t/\alpha)$$
$$w = \alpha \cosh(t/\alpha) \cos\chi$$
$$x = \alpha \cosh(t/\alpha) \sin\chi \cos\theta$$
$$y = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi$$
$$w = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi$$

$\therefore \; ds^2 = -dt^2 + \alpha^2 \cosh(t/\alpha) \Big[\underbrace{d\chi^2 + \sin^2\chi \, (\underbrace{d\theta^2 + \sin^2\theta \, d\phi^2}_{\text{2 sphere}})}_{\text{3 sphere}} \Big]$

$\mathbb{R} \times S^3$ topology

$\therefore$ de Sitter space shrinks to minimum at $t=0$ then re-expands

using coord transformations to yield a static universe

$$d\bar{s}^2 = -(dt')^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2$$

$$t' \in -\frac{\pi}{2}, \frac{\pi}{2}$$

$$\chi \in 0, \pi$$

if $R < 0$ anti-de Sitter space (AdS)

$$ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$$

hyperboloid w/ $-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2$

$$u = \alpha \sin(t') \cosh(\rho)$$
$$v = \alpha \cos(t') \cosh(\rho)$$
$$x = \alpha \sinh(\rho) \cos\theta$$
$$y = \alpha \sinh(\rho) \sin\theta \cos\phi$$
$$z = \alpha \sinh(\rho) \sin\theta \sin\phi$$

$$\Rightarrow ds^2 = \alpha^2\left(-\cosh^2(\rho)\,dt'^2 + d\rho^2 + \sinh^2(\rho)\,d\Omega_2^2\right)$$

Note; $t' = t' + 2\pi$

see conformal diagrams of de Sitter and anti de Sitter spacetimes on p 325 + 327

AdS/CFT correspondence

CFT - conformally-invariant field theory

"... in a certain limit there is an equivalence between quantum gravity on an AdS background and a conformally-invariant nongravitational field theory defined on the boundary."

in 4-D ; $R_{\mu\nu} = \frac{R}{4} g_{\mu\nu}$

$$\therefore\ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{R}{4} g_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow T_{\mu\nu} = -\frac{R}{32\pi G} g_{\mu\nu}$$

$$\Rightarrow \rho = -p = \frac{R}{32\pi G}$$ - vacuum energy or cosmological constant

ordinary matter and radiation make the max symmetric solutions somewhat unrealistic so we have the RW metrics "the universe can be foliated into spacelike slices such that each 3-D slice is maximally symmetric,"

$\therefore$ our universe is $\mathbb{R} \times \Sigma$

$$\Rightarrow \quad ds^2 = -dt^2 + R^2(t)\, d\sigma^2$$

scale factor

$$d\sigma^2 = \gamma_{ij}\, du^i du^j$$

comoving coordinates - no $dt\,du^i$ - terms are absent

an observer at constant u^i is "comoving"

$\therefore$ only a comoving observer sees the universe as isotropic

$$^{(3)}R_{ijkl} = \frac{^{(3)}R}{6}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$$

$$^{(3)}R_{jl} = 2k\cdot\gamma_{jl} = \frac{^{(3)}R}{3}\gamma_{jl} \qquad k = {}^{(3)}R/6$$

Maximally Symmetric in 3D $\Rightarrow$ spherical symmetry

$$\therefore\ d\sigma^2 = \gamma_{ij}\, du^i du^j = e^{2\beta(\bar{r})} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

$$^{(3)}R_{11} = \frac{2}{\bar{r}}\partial_1\beta$$

$$^{(3)}R_{22} = e^{-2\beta}(\bar{r}\partial_1\beta - 1) + 1$$

$$^{(3)}R_{33} = [e^{-2\beta}(\bar{r}\partial_1\beta - 1) + 1]\sin^2\theta$$

$$\beta = -\frac{1}{2}\ln(1 - k\bar{r}^2)$$

$$\Rightarrow d\sigma^2 = \frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega^2$$

Normalize k $\Rightarrow$ $k \in \{+1, 0, -1\}$

$k = -1$ - negative curvature (open univ)

$k = 0$ - no curvature (flat univ)

$k = +1$ - positive curvature (closed univ)

use $\quad d\chi = \frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}} \qquad \bar{r} = S_k(\chi)$

where $\quad S_k(\chi) \equiv \begin{cases} \sin(\chi) & k = +1 \\ \chi & k = 0 \\ \sinh(\chi) & k = -1 \end{cases}$

$$\therefore \quad d\sigma^2 = d\chi^2 + S_k^2(\chi)\, d\Omega^2$$

$k = 0 \Rightarrow d\sigma^2 = d\chi^2 + \chi^2 d\Omega^2 \qquad \chi \in -\infty, \infty$

$k = +1 \Rightarrow d\sigma^2 = d\chi^2 + \sin^2\chi\, d\Omega^2 \qquad \chi \in 0, 2\pi$

$k = -1 \Rightarrow d\sigma^2 = d\chi^2 + \sinh^2\chi\, d\Omega^2 \qquad \chi \in -\infty, \infty$

$$\therefore \quad ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega^2\right]$$

Robertson-Walker (RW) metric

RW is invariant under $R \to \lambda^{-1} R$

$$\bar{r} \to \lambda \bar{r}$$

$$k \to \lambda^{-2} k$$

$\therefore$ k is normalized to $\{+1, 0, -1\}$

dimensionless scale factor $a(t) = \dfrac{R(t)}{R_0}$

$$r = R_0 \bar{r} \qquad \kappa = \frac{k}{R_0^2}$$

$$\therefore \quad ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]$$

using $\acute{a} \equiv \dfrac{da}{dt}$ the Christoffel symbols are

$$\Gamma^0_{11} = \frac{a\acute{a}}{1-\kappa r^2} \quad \Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2} \quad \Gamma^0_{22} = a\acute{a}r^2 \quad \Gamma^0_{33} = a\acute{a}r^2 \sin^2\theta$$

$$\Gamma^1_{01} = \Gamma^2_{02} \quad \Gamma^3_{03} = \frac{\acute{a}}{a} \quad \Gamma^1_{22} = -r(1-\kappa r^2) \quad \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^1_{33} = -r(1-\kappa r^2)\sin^2\theta \quad \Gamma^2_{33} = -\sin\theta\cos\theta \quad \Gamma^3_{23} = \cot\theta$$

the Ricci tensor is

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad R_{11} = \frac{a\ddot{a} + 2\acute{a}^2 + 2\kappa}{1-\kappa r^2}$$

$$R_{22} = r^2(a\ddot{a} + 2\acute{a}^2 + 2\kappa) \quad R_{33} = r^2(a\ddot{a} + 2\acute{a}^2 + 2\kappa)\sin^2\theta$$

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\acute{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]$$

Next plug RW metric into Einstein's Eqn to derive the Friedmann eqns relating scale factor to energy/momentum

$$U^\mu = (1,0,0,0) \quad \text{- fluid at rest in comoving coords}$$

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$$

$$\Rightarrow T = -\rho + 3p$$

$$0 = \nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda$$

$$= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p)$$

Eqn of State (EOS) $\quad p = w\rho$

$$\Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \quad \therefore \rho \propto a^{-3(1+w)}$$

NDEC $\Rightarrow |w| \leq 1$

Matter-dominated Universe $\Rightarrow p \approx 0, \; w = 0 \quad \rho \propto a^{-3}$

for EM $\quad T^{\mu\nu} = F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}$

$$T^\mu{}_\mu = F^{\mu\lambda} F_{\mu\lambda} - \frac{1}{4}(4) F^{\lambda\sigma} F_{\lambda\sigma} = 0$$

Radiation-dominated Universe $\therefore \rho = 3p, \; w = \frac{1}{3} \quad \rho \propto a^{-4}$

$\rho \propto a^{-3}$ - energy decreases as volume increases

$\rho \propto a^{-4}$ - same as matter but photons are also redshifted as Univ expands

vacuum energy dominated $\rho = -p$, $w = -1$ $\rho \propto a^0$

describes de Sitter and anti de Sitter space

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

$\mu\nu = 00$ $$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p)$$

$\mu\nu = ij$ $$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

Friedmann Eqns

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

Friedmann Robertson Walker Metric (FRW)

$H = \frac{\dot{a}}{a}$ - Hubble parameter

currently $H = 70 \pm 10 \ \frac{km/s}{Mpc}$

$d_H = H_0^{-1} c$ - Hubble length

$t_H = H_0^{-1}$ - Hubble time

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \text{ - deceleration parameter}$$

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}} \text{ - density parameter}$$

$$\rho_{crit} = \frac{3H^2}{8\pi G} \text{ - critical density}$$

$$\Omega - 1 = \frac{k}{H^2a^2} \text{ - rewriting of Friedmann Eqn}$$

$$\rho < \rho_{crit} \Rightarrow \Omega < 1 \Rightarrow k < 0 \text{ - open}$$

$$\rho = \rho_{crit} \Rightarrow \Omega = 1 \Rightarrow k = 0 \text{ - flat}$$

$$\rho > \rho_{crit} \Rightarrow \Omega > 1 \Rightarrow k > 0 \text{ - closed}$$

imagine different components of energy density evolve as power laws

$$\rho_i = \rho_{i0}\, a^{-n_i}$$

$$\Rightarrow w_i = \tfrac{1}{3} n_i - 1$$

$$\rho_c \equiv -\frac{3k}{8\pi G a^2} \qquad \Omega_c = -\frac{k}{H^2a^2}$$

	w_i	n_i
matter	0	3
radiation	$\frac{1}{3}$	4
curvature	$-\frac{1}{3}$	2
vacuum	-1	0

$$H^2 = \frac{8\pi G}{3}\sum_{i(c)} \rho_i$$

$$1 = \sum_{i(c)} \Omega_i \quad \therefore \quad \Omega_c = 1 - \Omega$$

curvature density parameter; total density parameter

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G \sum_{i(c)} (1 + w_i)\rho_i$$

bc $|w_i| \leq 1$ if all $\rho_i \geq 0$ $\dot{H} \leq 0$

expansion of the universe should slow down

ρ_c can be negative - from vacuum energy (or positive)

for 1 type of energy density

$$\rho \propto a^{-n} \Rightarrow \dot{a} \propto a^{1-n/2}$$

$$\therefore a \propto t^{2/n} \quad \text{matter } a \propto t^{2/3}$$

$$\text{radiation } a \propto t^{1/2}$$

Big Bang universe started as singularity @ $t = 0$

for vacuum energy: $n = 0$

$$\therefore ds^2 = -dt^2 + e^{Ht}\left[dx^2 + dy^2 + dz^2\right]$$

space expands exponentially

for the universe to collapse H passes through 0 @ a_*

$$H^2 = 0 = \frac{8\pi G}{3}\left(\rho_{M0} a_*^{-3} + \rho_{\Lambda 0} + \rho_{c0} a_*^{-2}\right)$$

$$\Rightarrow \Omega_{\Lambda 0} a_*^3 + (1 - \Omega_{M0} - \Omega_\Lambda) a_* + \Omega_{M0} = 0$$

if $\Omega_{\Lambda 0} = 0$ and $\Omega_{M0} \leq 1$ - expand forever

if $\Omega_{\Lambda 0} = 0$ and $\Omega_{M0} > 1$ - recollapse

currently $\Omega_{M0} \sim 0.3$ $\Omega_{\Lambda 0} \sim 0.7$

for a static universe $\dot{a} = 0$ $\ddot{a} = 0$

$$\therefore p = -\frac{1}{3}\rho \qquad \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

$\Rightarrow$ we also need vacuum energy

$\rho_\Lambda = \frac{1}{2}\rho_M$ - Einstein static Universe

Redshift $\quad \dfrac{\omega_{obs}}{\omega_{em}} = \dfrac{a_{em}}{a_{obs}}$

$$z_{em} = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

$$a_{em} = \frac{1}{1 + z_{em}}$$

Relates redshift to scale factor

observed velocity of the expanding universe

$$v = \dot{d}_p = \dot{a} R_0 \chi = \frac{\dot{a}}{a} d_p$$

$$v = H_0 d_p$$ - Hubble law

$$d_L^2 = \frac{L}{4\pi F}$$ - luminosity distance

L - absolute luminosity

F - flux measured by the observer

$$F/L = 1/A(d) = 1/4\pi d^2$$

$$\therefore F/L = \frac{1}{(1+z)^2 A} \qquad A = 4\pi R_0^2 S_k^2(\chi)$$

$$d_L = (1+z) R_0 S_k(\chi)$$

$$d_M = \frac{u}{\dot{\theta}}$$ - proper motion distance

$$d_A = \frac{R}{\theta}$$ - angular diameter distance

$$d_L = (1+z)\, d_M = (1+z)^2 d_A$$