Perturbation Theory and Gravitational Radiation

assume a weak field that can vary w/ time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

linearized GR

$$h_{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} h_{\mu\nu}$$ — Lorentz transform of perturbation

$$\Rightarrow \Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu})$$

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho}$$

$$= \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma})$$

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^{\sigma}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}_{\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu})$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$

$$\therefore R = \partial_\mu\partial_\nu h^{\mu\nu} - h$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$$

$$= \frac{1}{2}(\partial_\sigma\partial_\nu h^{\sigma}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}_{\nu} - \partial_\mu\partial_\nu h$$

$$- \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h)$$

can also be derived by varying $\mathcal{L}$ wrt $\delta h_{\mu\nu}$

we can use $\partial_\mu T^{\mu\nu} = 0$ because the lowest nonvanishing order of $T_{\mu\nu}$ is the same order as the perturbation

$\mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$ – Lie derivative of the metric along ξ_μ

parameter of perturbation

$\Rightarrow h_{\mu\nu}^{(\epsilon)} = h_{\mu\nu} + 2\epsilon\, \partial_{(\mu}\xi_{\nu)}$ ← gauge transformation

note: $R_{\mu\nu\rho\sigma}(h_{\mu\nu}^{(\epsilon_1)}) = R_{\mu\nu\rho\sigma}(h_{\mu\nu}^{(\epsilon_2)})$ (7.15)

the Riemann tensor is invariant wrt gauge transformations

∴ gravity is unaffected by gauge transformations

$h_{\mu\nu}$ can be decomposed as: $h_{00} = -2\Phi$ – scalar

$h_{0i} = w_i$ – vector

$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$ – tensor

$\Psi = -\frac{1}{6}\delta^{ij}h_{ij}$ – trace of h_{ij}

strain → $s_{ij} = \frac{1}{2}(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij})$ – trace-free

$$\Rightarrow ds^2 = -(1+2\Phi)dt^2 + w_i(dt\,dx^i + dx^i dt) + [(1-2\Psi)\delta_{ij} + 2s_{ij}]dx^i dx^j$$

$\therefore \Gamma^0_{00} = \partial_0\Phi$ $\qquad \Gamma^i_{j0} = \partial_{[j}w_{i]} + \frac{1}{2}\partial_0 h_{ij}$

$\Gamma^i_{00} = \partial_i\Phi + \partial_0 w_i$ $\qquad \Gamma^0_{jk} = -\partial_{(j}w_{k)} + \frac{1}{2}\partial_0 h_{jk}$

$\Gamma^0_{j0} = \partial_j\Phi$ $\qquad \Gamma^i_{jk} = \partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}$

$$p^\mu = \frac{dx^\mu}{d\lambda} \qquad v^i = \frac{dx^i}{dt}$$

$$p^0 = \frac{dt}{d\lambda} = E \ , \ p^i = E v^i$$

geodesic eqn: $\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = 0$

$$\Rightarrow \frac{dp^\mu}{dt} = -\Gamma^\mu_{\rho\sigma} \frac{p^\rho p^\sigma}{E}$$

$$\mu = 0 \Rightarrow \frac{dE}{dt} = -E\left[\partial_0 \Phi + 2(\partial_k \Phi) v^k - \left(\partial_{(j} w_{k)} - \tfrac{1}{2}\partial_0 h_{jk}\right) v^j v^k\right]$$

$$\mu = i \Rightarrow \frac{dp^i}{dt} = -E\Big[\partial_i \Phi + \partial_0 w_i + 2\left(\partial_{[i} w_{j]} + \partial_0 h_{ij}\right) v^j$$
$$+ \left(\partial_{(j} h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right) v^j v^k\Big]$$

gravito-electric field: $G^i \equiv -\partial_i \Phi - \partial_0 w_i$

gravito-magnetic field: $H^i \equiv (\nabla \times \vec{w})^i = \epsilon^{ijk}\partial_j w_k$

$$\therefore \frac{dp^i}{dt} = E\left[G^i + (\vec{v} \times H)^i - 2(\partial_0 h_{ij}) v^j - \left(\partial_{(j} h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right) v^j v^k\right]$$

how particles respond
to scalar + vector
perturbations

Linearized Einstein Eqns

$$G_{00} = 2\nabla^2 \Psi + \partial_k \partial_l s^{kl}$$

$$G_{0j} = -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j \partial_k w^k + 2\partial_0 \partial_j \Psi + \partial_0 \partial_k s_j{}^k$$

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i \partial_j)(\Phi - \Psi) + \delta_{ij}\partial_0 \partial_k w^k - \partial_0 \partial_{(i} w_{j)}$$
$$+ 2\delta_{ij}\partial_0^2 \Psi - \Box s_{ij} + 2\partial_k \partial_{(i} s_{j)}{}^k - \delta_{ij}\partial_k \partial_l s^{kl}$$

Note that using $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ reveals that most of the metric components are constraints not degrees of freedom

example: $G_{00} = 8\pi G T_{00}$

$$\Rightarrow \nabla^2 \Psi = 4\pi G T_{00} - \frac{1}{2}\partial_k \partial_l s^{kl} \quad \text{— defines } \Psi$$
(w/o time derivatives)

$$G_{0j} = 8\pi G T_{0j}$$

$$\Rightarrow (\delta_{jk}\nabla^2 - \partial_j \partial_k) w^k = -16\pi G T_{0j} + 4\partial_0 \partial_j \Psi + 2\partial_0 \partial_k s_j{}^k$$
defines w^k (w/o time derivatives)

$$G_{ij} = 8\pi G T_{ij}$$
defines Φ (w/o time derivatives)

$$\Rightarrow (\delta_{ij}\nabla^2 - \partial_i \partial_j)\Phi = 8\pi G T_{ij} + (\delta_{ij}\nabla^2 - \partial_i\partial_j - 2\delta_{ij}\partial_0^2)\Psi$$
$$- \delta_{ij}\partial_0 \partial_k w^k + \partial_0 \partial_{(i} w_{j)} + \Box s_{ij} - 2\partial_k \partial_{(i} s_{j)}{}^k - \delta_{ij}\partial_k \partial_l s^{kl}$$

only the strain s_{ij} has degrees of freedom

Gauge choices:
$$\Phi \to \Phi + \partial_0 \xi^0$$
$$w_i \to w_i + \partial_0 \xi^i - \partial_i \xi^0$$
$$\Psi \to \Psi - \frac{1}{3}\partial_i \xi^i$$
$$s_{ij} \to s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k \xi^k \delta_{ij}$$

transverse gauge: $\partial_i s^{ij} = 0$

$$\therefore \nabla^2 \xi^j + \frac{1}{3}\partial_j \partial_i \xi^i = -2\partial_i s^{ij}$$

to determine ξ^0 ; $\partial_i w^i = 0$

$$\therefore \nabla^2 \xi^0 = \partial_i w^i + \partial_0 \partial_i \xi^i$$

in this gauge: $G_{00} = 2\nabla^2\Psi = 8\pi G T_{00}$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi = 8\pi G T_{0j}$$

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} = 8\pi G T_{ij}$$

Synchronous gauge : $\Phi = 0$

$$\therefore \partial_0 \xi^0 = -\Phi$$

$$w^i = 0$$

$$\therefore \partial_0 \xi^i = -w^i + \partial_i \xi^0$$

$$\Rightarrow ds^2 = -dt^2 + (\delta_{ij} + h_{ij})\,dx^i dx^j$$

Gravitational Lensing

Newtonian limit w/ static sources but test particles that move at any velocity

use dust — $T_{\mu\nu} = \rho U_\mu U_\nu = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

transverse gauge $\Rightarrow$ $\nabla^2\Psi = 4\pi G\rho$

$$\nabla^2 w_j = 0$$

$$(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \nabla^2 s_{ij} = 0$$

solutions should be nonsingular and well-behaved at infinity

$\therefore\ w^i = 0 \qquad 2\nabla^2(\Phi - \Psi) = 0 \quad \therefore\ \Phi = \Psi$

$\nabla^2 s_{ij} = 0 \quad \therefore\ s_{ij} = 0$

$$\Rightarrow ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)$$

$$h_{\mu\nu} = -2\Phi\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \nabla^2\Phi = 4\pi G\rho$$

Geodesic: $x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)$

background geodesic solution

assume $x^{(1)i}\partial_i\Phi \ll \Phi$ ← potential varies only slighty along true geodesic

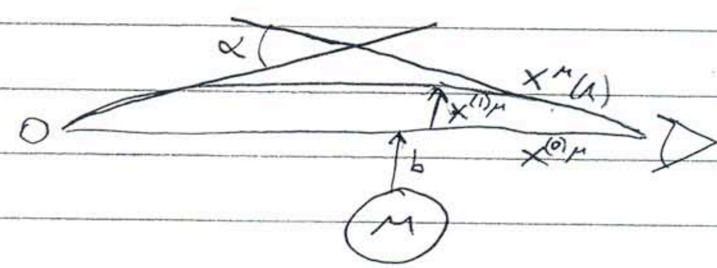

Fig 7.3 p.288

$$k^\mu \equiv \frac{dx^{(0)\mu}}{d\lambda}, \quad l^\mu \equiv \frac{dx^{(1)\mu}}{d\lambda} \qquad g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

← null vector

@ 0^{th} order $\eta_{\mu\nu}k^\mu k^\nu = 0 = -k_0k^0 + k_ik^i$

$$\therefore\ (k_0)^2 = (k_ik^i)^2 \equiv k^2$$

(A) 1^{st} order $\quad 2\eta_{\mu\nu}k^\mu l^\nu + h_{\mu\nu}k^\mu k^\nu = 0$

$$\therefore\ -kl^0 + l_i k^i = 2k^2\Phi$$

$$\Gamma^0_{0i} = \Gamma^i_{00} = \partial_i\Phi$$

$$\Gamma^i_{jk} = \delta_{jk}\partial_i\Phi - \delta_{ik}\partial_j\Phi - \delta_{ij}\partial_k\Phi$$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

$$\Rightarrow \frac{dl^\mu}{d\lambda} = -\Gamma^\mu_{\rho\sigma}k^\rho k^\sigma, \quad \frac{dl^0}{d\lambda} = -2k(k_i\nabla^i\Phi)$$

$$\frac{dl^i}{d\lambda} = -2k^2\left(\nabla^i\Phi - k^{-2}(k_j\nabla^j\Phi)k^i\right)$$

$$l^0 = \int\frac{dl^0}{d\lambda}d\lambda = -2k\Phi$$

$$\therefore\ l_i k^i = kl^0 + 2k^2\Phi = 0 \qquad \vec{l}\perp\vec{k}$$

$$\alpha = -\frac{\Delta\vec{l}}{k} \qquad \Delta\vec{l} = \int\frac{dl}{d\lambda}d\lambda = -2k^2\int\vec{\nabla}_\perp\Phi\, d\lambda$$

$$\vec{\nabla}_\perp\Phi = \vec{\nabla}\Phi - \vec{\nabla}_\parallel\Phi$$
$$= \vec{\nabla}\Phi - k^{-2}(\vec{k}\cdot\vec{\nabla}\Phi)\vec{k}$$

$$\therefore\ \alpha = 2\int\vec{\nabla}_\perp\Phi\, ds$$

if $\Phi = -\frac{GM}{r} = -\frac{GM}{(b^2+x^2)^{\frac{1}{2}}}$

$\Rightarrow \alpha = \frac{4GM}{b}$ ← Deflection of light by gravity
GR's 1^{st} test

Gravitational time delay ; $t = \int \frac{dx^0}{d\lambda} d\lambda$

$$\Delta t = \int \frac{dx^{(1)0}}{d\lambda} d\lambda = \int l^0 d\lambda$$

$$= -2k \int \Phi \, d\lambda = -2 \int \Phi \, ds$$

similar to space having an index of refraction $n = 1 - 2\Phi$

Gravitational Waves

set $T_{\mu\nu} = 0$ - far from sources

$$\Rightarrow \nabla^2 \Psi = 0 \quad \therefore \Psi = 0$$

$$\nabla^2 w_j = 0 \quad \therefore w_j = 0$$

$$\nabla^2 \Phi = 0 \quad \therefore \Phi = 0$$

$\Box s_{ij} = 0$ - wave eqn

Transverse Traceless gauge

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{pmatrix}$$

$$\therefore \Box h_{\mu\nu}^{TT} = 0 = \partial^\sigma \partial_\sigma h_{\mu\nu} = 0$$

$\Rightarrow h_{\mu\nu}^{TT} = C_{\mu\nu} e^{ik_\sigma x^\sigma} \leftarrow$ Plane Waves

$\nwarrow$ constant symmetric tensor

$$0 = \Box h^{TT}_{\mu\nu}$$
$$= \eta^{\rho\sigma} \partial_\rho \partial_\sigma h^{TT}_{\mu\nu}$$
$$= \eta^{\rho\sigma} \partial_\rho (i k_\sigma h^{TT}_{\mu\nu})$$
$$= - \eta^{\rho\sigma} k_\rho k_\sigma h^{TT}_{\mu\nu}$$
$$= - k_\sigma k^\sigma h^{TT}_{\mu\nu} \quad \therefore \quad k_\sigma k^\sigma = 0$$

Plane Wave solution works for a null wave vector
$\therefore$ GW's move at (or close to) the speed of light

$$k^\sigma = (\omega, k^1, k^2, k^3) \quad \therefore \quad \omega^2 = \delta_{ij} k^i k^j$$

for a linearized solution, we can superimpose waves

Transverse implies $k_\mu h^{TT\mu\nu} = 0 \quad \therefore \quad k_\mu C^{\mu\nu} = 0$

$$\Rightarrow \text{ for } k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega)$$

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for particles w/ a 4-velocity $U^\mu(x)$ + seperation S^μ

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu_{\ \nu\rho\sigma} U^\nu U^\rho S^\sigma$$

$U^{\nu} \approx (1,0,0,0)$

$R_{\mu 0 0 \sigma} \Rightarrow \frac{1}{2} \partial_0 \partial_0 h^{TT}_{\mu\sigma}$

Geodesic Deviation Eqn $\Rightarrow \frac{\partial^2}{\partial t^2} S^{\mu} = \frac{1}{2} S^{\sigma} \frac{\partial^2}{\partial t^2} h^{TT\mu}{}_{\sigma}$

if wave travels in z-direction, particles move only in x + y - directions

$$h_+ = C_{11} \quad h_\times = C_{12} \quad \therefore \; C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

if $h_\times = 0 \Rightarrow \frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} \left(h_+ e^{i k_\sigma x^\sigma} \right)$

$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} \left(h_+ e^{i k_\sigma x^\sigma} \right)$$

$$\therefore \; S^1 = \left(1 + \frac{1}{2} h_+ e^{i k_\sigma x^\sigma} \right) S^1(0)$$

$$S^2 = \left(1 - \frac{1}{2} h_+ e^{i k_\sigma x^\sigma} \right) S^2(0)$$

if $h_+ = 0 \Rightarrow S^1 = S^1(0) + \frac{1}{2} h_\times e^{i k_\sigma x^\sigma} S^2(0)$

$$S^2 = S^2(0) + \frac{1}{2} h_\times e^{i k_\sigma x^\sigma} S^1(0)$$

see Fig 7.4 p 297
+ Fig 7.5 p 298

2 - polarizations $+$ and $\times$

$$h_R = \frac{1}{\sqrt{2}}(h_+ + i h_\times) \quad \text{- right circularly polarized}$$

$$h_L = \frac{1}{\sqrt{2}}(h_+ - i h_+) \quad \text{- left circularly polarized}$$

see Fig 7.6 p 298

GW's correspond to spin $\left[S = \frac{360^\circ}{\theta}\right]$ 2 particles (gravitons)

nonlinearity - $h_{\mu\nu}$ should couple to its own $T_{\mu\nu}$ and that of the matter field resulting in nonlinearities $\rightarrow$ GR

Sources: solve $\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ - trace-reversed perturbation

use Green function $G(x^\sigma - y^\sigma)$

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma)$$

$$\therefore \bar{h}_{\mu\nu}(x^\sigma) = -16\pi G \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) d^4 y$$

$$\Rightarrow \bar{h}_{ij}(t; \vec{x}) = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r) \Leftarrow \text{quadruple formula}$$

$$I_{ij}(t) = \int y^i y^j T^{00}(t, \vec{y}) d^3 y \Leftarrow \text{quadruple momentum tensor}$$