$$\Rightarrow J = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu R^\nu$$

ang momentum

— stop

Charged Black Holes (Reissner - Nordström)

same symmetry as Schwarzschild

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2$$

but the RHS is non-zero because of the EM Field

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

general solutions

$$F_{tr} = f(r,t) = -F_{rt} \qquad F_{\theta\phi} = g(r,t)\sin\theta = -F_{\phi\theta}$$

else zero.

F_{tr} — radial E-field

$F_{\theta\phi}$ - radial B-field

$$g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0$$

$$\nabla_{[\mu}F_{\nu\rho]} = 0$$

$\Rightarrow$ Maxwell's eqns + Einstien's eqns

$F_{\nu\sigma}$ enter's Einstein's Eqns through $T_{\mu\nu}$

$g_{\mu\nu}$ enters Maxwell's Eqns

⇒ Reissner - Nordström Metric

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$$

$$\Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}$$

M - mass
Q - electric charge
P - magnetic charge

$$E_r = F_{rt} = \frac{Q}{r^2}$$

$$B_r = \frac{F_{\theta\phi}}{r^2 \sin\theta} = \frac{P}{r^2}$$

if $g^{rr} = 0$ represents the event horizon

$$g^{rr}(r) = \Delta(r) = 1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2} = 0$$

$$r_\pm = GM \pm \sqrt{G^2M^2 - G(Q^2+P^2)}$$

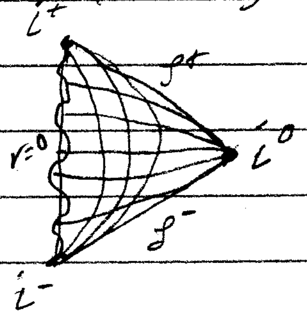
∴ there may be 0, 1 or 2 event horizons depending on how GM^2 relates to Q^2+P^2

0 horizons ; $GM^2 < Q^2 + P^2$

- Δ is always positive
- t is always timelike
- r is always spacelike
- $r=0$ is still singular (but a timelike line)
- an observer can travel to and from the singularity safely (naked singularity)

• geodesics imply a repulsive singularity
timelike geodesics never intersect $r=0$
the approach and then move away

Conformal diagram



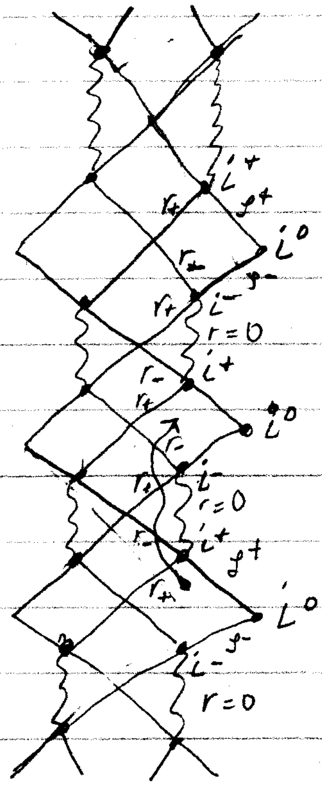
$GM^2 < Q^2 + P^2$ is impossible since it would never collapse in the first place $\therefore$ unphysical

2 horizons ; $GM^2 > Q^2 + P^2$

EM Field energy is less than gravitational energy so collapse is possible

• coord singularities at r_+ & r_-
• $r_\pm$ are both null surfaces and event horizons
• $r=0$ is a timelike line and singularity
• r_+ is just like $2GM$ in the Schwarzschild Metric
at r_+ r switches from spacelike to timelike
- at $r=r_-$, r switches back to spacelike and motion towards the singularity can be arrested and an observer can move backwards out past r_+ (out a different black hole)

conformal diagram for Reissner-Nordstrom



An observer about to pass r_- will see the entire infinite history of the outside universe in a finite amount of proper time infinitely blueshifted.

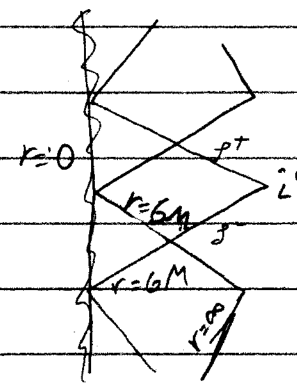
Any nonspherically symmetric perturbation that comes into an RN black hole will disturb the geometry of the spacetime therefore possibly destroying the network of worm holes or neutralizing the black hole.

1 singularity : $GM^2 = Q^2 + P^2$

- extreme Reissner-Nordström solution

- $\Delta(r) = 0$ where $r = GM$
- $r = GM$ becomes null but is timelike on both sides
- $r = 0$ is a timelike line

conformal diagram



similar to 2 horizon case but singularity is always on one side

since charge and mass balance two identical extreme RN black holes will attract gravitationally but repel electromagnetically

RN metric w/ $GM^2 = Q^2$

$$ds^2 = -\left(1 - \frac{GM}{r}\right)^2 dt^2 + \left(1 - \frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega^2$$

using $\rho = r - GM$

$$ds^2 = -H^{-2}(\rho)\, dt^2 + H^2(\rho)\left[d\rho^2 + \rho^2 d\Omega^2\right]$$

$$H(\rho) = 1 + \frac{GM}{\rho}$$

$$\Rightarrow ds^2 = -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})[dx^2 + dy^2 + dz^2]$$

$$H = 1 + \frac{GM}{|\vec{x}|}$$

$$\vec{E}_r = F_{rt} = \frac{Q}{r^2} = \partial_r A_0 \quad \text{- electric field}$$

$$\therefore A_0 = \frac{-Q}{r} \quad \text{- scalar potential}$$

$$\Rightarrow A_0 = -\frac{\sqrt{G}M}{\rho + GM} \rightarrow \sqrt{G}A_0 = H^{-1} - 1$$

plugging the metric and electrostatic potential into Einstein and Maxwell's eqns (assuming H is time indepent) gives

$$\nabla^2 H = 0$$

at infinity $\quad H = 1 + \sum_{a=1}^{N} \frac{GM_a}{|\vec{x} - \vec{x}_a|}$

describes a system of N extremal RN black holes with masses M_a and charges $Q_a = \sqrt{G}M_a$

Rotating (Kerr) Black Holes

- axial symmetry not spherical symmetry but also stationary (timelike Killing vector)

Kerr metric (1963)

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dt\,d\phi + d\phi\,dt)$$

$$+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

$$\Delta(r) = r^2 - 2GMr + a^2$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$$

$a = J/M$ - ang momentum per unit mass

Replacing $2GMr \rightarrow 2GMr - G(Q^2 + P^2)$ results in the Kerr-Newman (spinning charged black hole) metric

$$A_0 = \frac{Qr - Pa\cos\theta}{\rho^2} \qquad A_\phi = \frac{-Qar\sin^2\theta + P(r^2+a^2)\cos\theta}{\rho^2}$$

Boyer-Lindquist coordinates (t, r, θ, ϕ)

if a is fixed and $M \rightarrow 0$

$$ds^2 = -dt^2 + \frac{(r^2 + a^2\cos^2\theta)}{(r^2+a^2)}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2$$
$$+ (r^2+a^2)\sin^2\theta\, d\phi^2$$

$$\left.\begin{aligned} x &= (r^2+a^2)^{1/2}\sin\theta\cos\phi \\ y &= (r^2+a^2)^{1/2}\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \text{translation to cartesian}$$

2 Killing vectors $K = \partial_t$ $R = \partial_\phi$

K is not orthogonal to any hypersurfaces so the metric is stationary but not static

Killing tensor $\nabla_{(\lambda} \sigma_{\mu_1 \ldots \mu_n)} = 0$

$$\sigma_{\mu\nu} = 2\rho^2 l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}$$ – Kerr Killing tensor

$$l^\mu = \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a)$$

$$n^\mu = \frac{1}{2\rho^2}(r^2 + a^2, -\Delta, 0, a)$$

$$l^\mu l_\mu = 0 \;, \; n^\mu n_\mu = 0 \;, \; l^\mu n_\mu = -1$$

for event horizons $g^{rr} = \frac{\Delta}{\rho^2} = 0$

$$\therefore \; \Delta(r) = r^2 - 2GMr + a^2 = 0$$

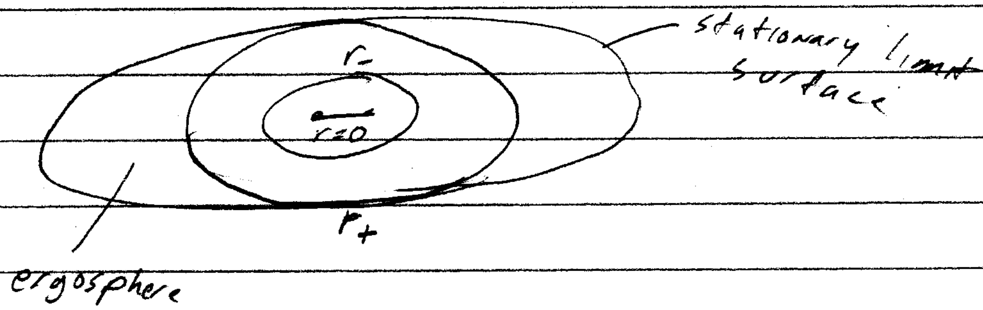
$$\Rightarrow r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$$

$GM < a$ – naked singularity (like RN)

$GM = a$ – extremal Kerr (unstable)

$GM > a$ – 2 horizons

both $r_\pm$ are event horizons but not Killing horizons



$$K^\mu K_\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2\theta) \neq 0 \ @ \ r_\pm$$

the Killing Vector is spacelike at r_+ except at the poles, where it is null

$K^\mu K_\mu = 0$ at the stationary limit surface

$$(r - GM)^2 = G^2M^2 - a^2\cos^2\theta$$

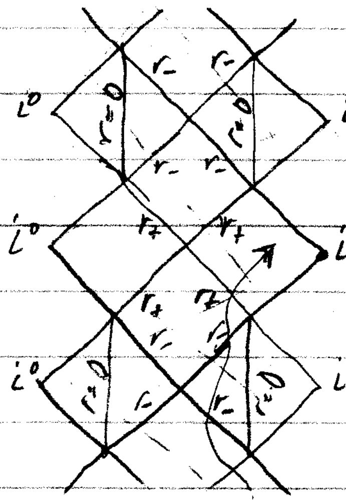
ergosphere - inside you must move w/ the black hole's rotation but you can move towards or away from the event horizon

singularity occurs at $\rho = 0$ not simply $r = 0$

since $\rho^2 = r^2 + a^2\cos^2\theta$ $\quad r = 0 + \theta = \frac{\pi}{2}$

this represents a ring instead of a pt.

you can pass through the ring to anther asymptotically flat spacetime where $r<0$, there are no horizons



Conformal Diagram for Kerr $G^2M^2 > a^2$

diff for diff values of θ

Near the ring we get closed timelike curves

if $\theta + t$ constant and r small and negative

$$ds^2 \approx a^2\left(1 + \frac{2GM}{r}\right)d\phi^2$$ - neg for small neg values of r

frame dragging - a photon emitted in the ϕ direction of a Kerr Black hole has

$$ds^2 = 0 = g_{tt}\,dt^2 + g_{t\phi}(dt\,d\phi + d\phi\,dt) + g_{\phi\phi}\,d\phi^2$$

$$\Rightarrow \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

$$\therefore \frac{d\phi}{dt} = 0, \quad \frac{d\phi}{dt} = \frac{a}{2G^2M^2 + a^2}$$

photon moving against hole

photon moving w/ hole

massive particles are dragged along w/ the hole's rotation inside the stationary limit

$$\Omega_H = \left(\frac{d\phi}{dt}\right)_-(r_+) = \frac{a}{r_+^2 + a^2}$$

minimum ang velocity of particle at the horizon

Black Hole Thermodynamics

motion along geodesics in the Kerr metric

$K = \partial_t$, $R = \partial_\phi$ - Killing vectors

$p^\mu = m \frac{dx^\mu}{d\tau}$ - 4 momentum

energy/mass

$$E = -K_\mu p^\mu = m\left(1 - \frac{2GMr}{\rho^2}\right)\frac{dt}{d\tau} + \frac{2mGMar}{\rho^2}\sin^2\theta \frac{d\phi}{d\tau}$$

ang mom/mass

$$L = R_\mu p^\mu = -\frac{2mGMar}{\rho^2}\sin^2\theta\frac{dt}{d\tau} + \frac{m(r^2+a^2)^2 - m\Delta a^2 \sin^2\theta}{\rho^2}\sin^2\theta\frac{d\phi}{d\tau}$$

inside the ergosphere K^μ is spacelike $\therefore E = -K_\mu p^\mu < 0$

$\therefore$ a particle inside the ergosphere either remains there or is accelerated until its energy is positive

Penrose process - method of extracting energy from a rotating black hole

- by throwing mass into the horizon, an object can excape the ergosphere with more energy than it originally had

∴ extracting energy from the black hole and reducing its angular momentum

$$\chi^\mu = K^\mu + \Omega_H R^\mu$$

null at event horizon

$$p^{(2)\mu}\chi_\mu < 0$$ - momentum of object thrown into the horizon

$$L^{(2)} < \frac{E^{(2)}}{\Omega_H}$$ - ang mom must be negative

$$\therefore \ \delta M = E^{(2)}$$
$$\delta J = L^{(2)} < \frac{\delta M}{\Omega_H}$$

the area of an event horizon cannot decrease although the mass decreases and ang mom decreases the combination of the two must keep the area const.

$$r_+ = GM + \sqrt{G^2M^2 - a^2}$$

$$\gamma_{ij}\,dx^i dx^j = (r_+^2 + a^2\cos^2\theta)\,d\theta^2 + \left[\frac{(r_+^2+a^2)^2\sin^2\theta}{r_+^2+a^2\cos^2\theta}\right]d\phi^2$$

$$A = \int\sqrt{|\gamma|}\,d\theta\,d\phi = \int\sqrt{(r_+^2+a^2)^2\sin^2\theta}\,d\theta\,d\phi$$

$$\therefore \ A = 4\pi(r_+^2 + a^2)$$

irreducible mass - mass as calculated from the area of the black hole

$$M_{irr}^2 = \frac{A}{16\pi G^2} \Rightarrow \frac{1}{2}\left(M^2 + \sqrt{M^4 - (J/G)^2}\right)$$

$$\delta M_{irr} = \frac{a}{4GM_{irr}\sqrt{G^2M^2 - a^2}}\left(\Omega_H^{-1}\delta M - \delta J\right) > 0$$

$$M - M_{irr} = M - \frac{1}{\sqrt{2}}\left(M^2 + \sqrt{M^4 - (J/G)^2}\right)^{1/2}$$

Max amount of energy that can be extracted from a black hole before rotation is slowed to zero

$$\delta A = 8\pi G \frac{a}{\Omega_H\sqrt{G^2M^2 - a^2}}\left(\delta M - \Omega_H\delta J\right) \geq 0$$

$$\Rightarrow \delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J$$

$$\kappa = \frac{\sqrt{G^2M^2 - a^2}}{2GM\left(GM + \sqrt{G^2M^2 - a^2}\right)}$$

similar to $dE = TdS - p\,dV$

$E \leftrightarrow M \qquad S \leftrightarrow A/4G \qquad T \leftrightarrow \kappa/2\pi$

each law of thermodynamic corresponds to a law of black hole mechanics

- thermal equilibrium $\rightarrow$ stationary BH
- 0^{th} law: in thermal equil temperature is const $\rightarrow$ surface gravity is const on the horizon
- 1^{st} law: $dE = TdS - pdV \rightarrow \delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J$
- 2^{nd} law: S never decreases $\rightarrow$ area never decreases

$T = \frac{\kappa}{2\pi}$ - Hawking Black hole radiation temp

$S = A/4G$ - Black hole entropy

$\delta\left(S + \frac{A}{4G}\right) \geq 0$ - combined entropy of matter and black holes never decreases "generalized 2^{nd} law"