Event Horizons

event horizon - hypersurface separating spacetime points that are connected to infinity by a timelike path from those not connected

$\therefore$ the event horizon is a null hypersurface - the normal of the event horizon is a null vector

Σ - hypersurface
$f(x)$ - constant function on the hypersurface
$\partial_\mu f$ - gradient normal to Σ
$x^\mu(\lambda)$ - null geodesics $\rightarrow$ generators of the hypersurface
ξ^μ - tangent to the geodesics $\Rightarrow$ normal vectors to hypersurface

$$\xi^\mu = \frac{dx^\mu}{d\lambda} = h(x) g^{\mu\nu} \partial_\nu f$$

function to make the geodesics affinely parameterized

$$\xi_\mu \xi^\mu = 0 \qquad\qquad \xi^\mu \nabla_\mu \xi^\nu = 0$$

definition of null vector

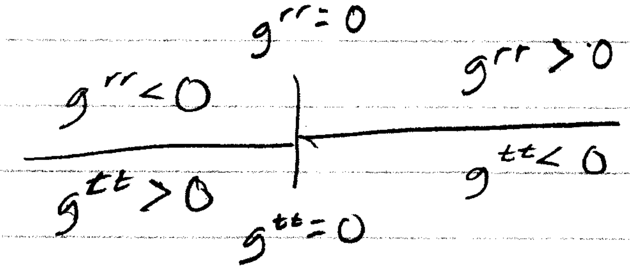
geodesic eqn

if hypersurfaces are null at a fixed radius r_H
r_H ~ event horizon boundary

$$g^{\mu\nu}(r_H)\, \partial_\mu r \;\; \partial_\nu r = 0 = g^{rr}(r_H)$$

one form normal to the hypersurface

at the event horizon g^{rr} switches from positive to negative



trapped surface – compact, spacelike, 2D submanifold where outgoing future-directed light rays converge in both directions on the submanifold

singularity theorems – once collapse reaches a certain point, evolution to a singularity is inevitable

example: "Let M be a manifold with generic metric $g_{\mu\nu}$, satisfying Einstein's equation with the strong energy condition imposed. If there is a trapped surface in M, there must be either a closed timelike curve or a singularity."

a generic metric implies that curvature can't consistently vanish in any direction

$\therefore$ singularities must exist in many GR solutions cosmic censorship conjecture - "Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominate energy condition."

naked singularities - singularities w/o horizons

Hawking's area theorem - "Assuming the weak condition and cosmic censorship, the area of a future event horizon in an asymptotically flat spacetime is nondecreasing."

Killing Horizons

Killing horizon - if a Killing vector field χ^μ is null along some null hypersurface Σ, Σ is a Killing horizon of χ^μ

- Every event horizon in a stationary, asymptotically flat spacetime is a Killing horizon
- If the spacetime is static χ^μ will be the Killing vector field $K^\mu = (\partial_t)^\mu$
- If stationary but not static, it will be axisymmetric $\therefore$ Killing vector field $\Rightarrow R^\mu = (\partial_\phi)^\mu$

$$\chi^\mu \propto K^\mu + \Omega_H R^\mu$$

surface gravity of a Killing horizon

at the Killing horizon:

surface gravity

$$\chi^\mu \nabla_\mu \chi^\nu = -\kappa \chi^\nu$$

integral curves of χ^μ may not be affinely parameterized

using $\nabla_{(\mu}\chi_{\nu)} = 0$ + $\chi_{[\mu}\nabla_\nu \chi_{\sigma]} = 0$

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)$$

normalize the time-translation Killing vector $K = \partial_t$

$$\rightarrow K_\mu K^\mu (r \rightarrow \infty) = -1$$

to fix the surface gravity of an associated Killing horizon

In a static, asymptotically flat spacetime, the surface gravity is the acceleration of a static observer near the horizon, as measured by a static observer at infinity

time-translation Killing field

$$K^\mu = V(x)\, U^\mu$$

4-velocity of a static observer

@ Killing horizon

$$V(x) = \sqrt{-K_\mu K^\mu} \qquad V(x) \in 0, 1$$

"redshift factor"

@∞

energy

4-momentum

freq

$$E = -p_\mu K^\mu \qquad \omega = -p_\mu U^\mu$$

$$\therefore \quad \omega = \frac{E}{V} \qquad \lambda = \frac{2\pi}{\omega}$$

$\lambda_2 = \frac{V_2}{V_1}\lambda_1$ ← wavelength observed by observer 2

$$a^\mu = U^\sigma \nabla_\sigma U^\mu \Rightarrow \quad a = \sqrt{a_\mu a^\mu} = V^{-1}\sqrt{\nabla_\mu V \nabla^\mu V}$$

$a \to \infty$ at the Killing horizon

$\therefore$ it will take an infinity acceleration to keep an object on a static trajectory

also $\quad K = Va = \sqrt{\nabla_\mu V \nabla^\mu V}$

An observer at infinity will detect the acceleration to be "redshifted" by a factor V

stationary limit surface – $K^\mu K_\mu = 0$ (ergosurface)
inside K^μ is spacelike and no observer can remain stationary

ergosphere – region between the stationary limit surface and the event horizon where timelike paths get dragged along w/ the black hole's rotation for a Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$K^\mu = (1,0,0,0) \qquad U^\mu = \left[\left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}}, 0, 0, 0\right]$$

$$\therefore \ V = \sqrt{1 - \frac{2GM}{r}}$$

$$a_\mu = \frac{GM}{r^2\left(1 - \frac{2GM}{r}\right)}\nabla_\mu r \rightarrow a = \frac{GM}{r^2\left(1 - \frac{2GM}{r}\right)^{1/2}}$$

$$\kappa = Va = \frac{GM}{r^2} \quad @\ r = 2GM \rightarrow \kappa = \frac{1}{4GM}$$

as M increases, the Schwarzschild radius increases $\therefore$ decreasing the surface gravity

Mass, Charge and Spin - characterize most stationary black-hole solutions

Charge: (Electric) $\quad \nabla_\nu F^{\mu\nu} = J_e^\mu$ - electric 4-current

$$Q = -\int_\Sigma d^3x \sqrt{|\gamma|}\, n_\mu J_e^\mu$$

positive charge density

$$= -\int_\Sigma d^3x \sqrt{|\gamma|}\, n_\mu \nabla_\nu F^{\mu\nu}$$

future-pointing normal vector

using Stoke's theorem:

$$Q = -\int_{\partial\Sigma} d^2x \sqrt{|\gamma^{(2)}|}\, n_\mu \sigma_\nu F^{\mu\nu}$$

σ^μ - outward-pointing normal vector

for the magnetic charge $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$
replaces $F^{\mu\nu}$

$\therefore$ if we know the EM field at spatial infinity ($F^{\mu\nu}$)
we know the total charge of the black hole

Energy (or Mass) ; $J_T^\mu = K_\nu T^{\mu\nu}$ - energy current

$$E_T = \int_\Sigma d^3x \sqrt{\gamma}\, n_\mu J_T^\mu$$

$$= \int_\Sigma d^3x \sqrt{\gamma}\, n_\mu K_\nu T^{\mu\nu} = 0 \text{ for Schwarzschild}$$

This can't be right

try $J_R^\mu = K_\nu R^{\mu\nu}$ $\quad K^\mu$ - timelike

$$\Rightarrow J_R^\mu = 8\pi G K_\nu \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu}\right)$$

$$\nabla_\mu J_R^\mu = \overset{\nearrow 0}{(\underbrace{\nabla_\mu K_\nu}_{\text{antisymmetric}})\underbrace{R^{\mu\nu}}_{\text{symmetric}}} + K_\nu (\nabla_\mu R^{\mu\nu})$$

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$$

$$\Rightarrow \nabla_\mu J_R^\mu = \frac{1}{2} K_\nu \nabla^\nu R = 0$$

R is const on the Killing Vector

$$\therefore E_R = \frac{1}{4\pi G}\int_\Sigma d^3x \sqrt{\gamma}\, n_\mu J^\mu_R$$

$$\nabla_\mu \nabla_\nu K^\mu = K^\mu R_{\mu\nu} \quad - 3.177$$

$$\therefore J^\mu_R = \nabla_\nu (\nabla^\mu K^\nu)$$

$$\Rightarrow E_R = \frac{1}{4\pi G}\int_\Sigma d^3x \sqrt{\gamma}\, n_\mu \nabla_\nu (\nabla^\mu K^\nu)$$

$$E_R = \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu K^\nu \leftarrow \text{Komar Integral}$$

for Schwartzschild: $n_\mu n^\mu = -1$, $\sigma_\mu \sigma^\mu = +1$

$$n_0 = -\left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}}, \quad \sigma_1 = \left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}}$$

else zero

$$\therefore n_\mu \sigma_\nu \nabla^\mu K^\nu = -\nabla^0 K^1$$

$$K^\mu = (1, 0, 0, 0)$$

$$\nabla^0 K^1 = g^{00} \nabla_0 K^1$$
$$= g^{00}(\partial_0 K^1 + \Gamma^1_{0\lambda} K^\lambda)$$

$$= g^{00} \Gamma^1_{00} K^0$$
$$= -\left(1 - \frac{2GM}{r}\right)^{-1} \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)$$
$$= -\frac{GM}{r^2}$$

$$\gamma^{(2)}_{ij} dx^i dx^j = r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \Rightarrow \sqrt{\gamma^{(2)}} = r^2 \sin\theta$$

$$\therefore \quad E_R = \frac{1}{4\pi G} \int d\theta\, d\phi\, r^2 \sin\theta \left(\frac{GM}{r^2}\right)$$

$$= M$$

Also, total energy can be defined using the generator of time translations in an asymptotically flat spacetime

ADM energy (Arnowitt, Deser and Misner)

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ← perturbation of flat spacetime

$h_{\mu\nu}$ may not be small everywhere, only at spatial infinity

$$E_{ADM} = \frac{1}{16\pi G} \int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}}\, \sigma^i \left(\partial_j h^j_{\ i} - \partial_i h^j_{\ j}\right)$$

if $h_{\mu\nu}$ is time-ind at infinity $\quad E_{ADM} = E_R$

positive energy theorem - "The ADM energy of a nonsingular, asymptotically flat spacetime obeying Einstein's Eqn and dominant energy condition is nonnegative. Furthermore, Minkowski is the only such spacetime with vanishing ADM energy".

Angular Momentum (Spin): $J^\mu_\phi = R_\nu R^{\mu\nu}$ - conserved current

$\uparrow$ rotating Killing vector

$$\Rightarrow J = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu R^\nu$$

ang momentum

Stop

Charged Black Holes (Reissner - Nordström)

same symmetry as Schwarzschild

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2$$

but the RHS is non-zero because of the EM Field

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

general solutions $\rightarrow$ $F_{tr} = f(r,t) = -F_{rt}$ $\qquad F_{\theta\phi} = g(r,t)\sin\theta = -F_{\phi\theta}$

else zero.

F_{tr} — radial E-field

$F_{\theta\phi}$ - radial B-field

$$g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0$$

$$\nabla_{[\mu}F_{\nu\rho]} = 0$$

Maxwell's eqns + Einstein's eqns

$F_{\nu\sigma}$ enter's Einstein's Eqns through $T_{\mu\nu}$

$g_{\mu\nu}$ enters Maxwell's Eqns