Schwarzschild Black Holes

Examine causal structure as defined by light cones
use radial null curves $ds^2 = 0 = d\theta^2 = d\phi^2$

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2$$

$$\therefore \frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}$$

as $r \to \infty$ $\frac{dt}{dr} \to \pm 1$ - normal light cone

as $r \to 2GM$ $\frac{dt}{dr} \to \pm\infty$ - light cone closes

light never "appears" to get to $r = 2GM$

as an infalling observer approaches $r = 2GM$ any fixed interval $\Delta\tau_1$ of their proper time corrisponds to a longer observered time interval $\Delta\tau_2$ - An outside observer sees someone approaching $r = 2GM$ as moving more and more slowly

"Does the observer reach $r = 2GM$ in a finite amount of proper time?"

replace t w/ a coord the moves slowly on null geodesics

$$t = \pm r^* + \text{const}$$

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \leftarrow \text{tortoise coordinate}$$

if $r \geq 2GM$ r^* is related to r in a physically relavent way

$$ds^2 = \left(1 - \frac{2GM}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2$$

as $r \to 2GM$ $g_{tt} = g_{r^*r^*} = 0$ but nothing is ∞
$r^* \to -\infty$

Eddington-Finkelstein coords
$v = t + r^*$ - infalling radial null geodesics
$u = t - r^*$ - outgoing radial null geodesics

$$\therefore ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + (dv\,dr + dr\,dv) + r^2 d\Omega^2$$

replace t w/ EF coords

$g = -r^4 \sin^2\theta$ - regular at $r = 2GM$
$\therefore$ metric is invertable

$$\frac{dv}{dr} = \begin{cases} 0 & \text{infalling EF} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{outgoing EF} \end{cases}$$

Fig 5.10 - light cones tilt as the pass $r = 2GM$
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Event Horizon - pt of no return to infinity

Black Hole - no light excapes the event horizon
- region separated from infinity by the event horizon

Maximally extended Schwarzschild solution

Note: as we decrease r along a radial null curve $v = \text{const}$

a local observer will not notice that he crossed the event horizon

In the (v,r) coord system we can cross the event horizon on future-directed paths but not past-directed

if we used u instead of v

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)du^2 - (du\,dr + dr\,du) + r^2 d\Omega^2$$

(As in Fig 5.11) this implies that we can only past through the event horizon on past-directed paths

to keep $v = \text{const}$ and decrease r $t \to +\infty$

to keep $u = \text{const}$ and decrease r $t \to -\infty$

$\therefore$ we have extended space-time into the future and past to cross the event horizon

Using both $u + v$ leads to

$$ds^2 = -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)(dv\,du + du\,dv) + r^2 d\Omega^2$$

where r is defined by

$$\frac{1}{2}(v-u) = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

Note: $r = 2GM$ implies either $v = -\infty$ or $u = +\infty$

a second coordinate change

$$v' = e^{v/4GM} = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r+t)/4GM}$$

$$u' = -e^{-u/4GM} = -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r-t)/4GM}$$

$$\therefore\ ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(dv'du' + du'dv') + r^2 d\Omega^2$$

v' + u' are null coords $\rightarrow$ $\partial/\partial v'$ + $\partial/\partial u'$ are null vectors

timelike vector $\rightarrow$ $T = \frac{1}{2}(v' + u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right)$

space like vector $\rightarrow$ $R = \frac{1}{2}(v' - u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right)$

$$\therefore\ ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dR^2) + r^2 d\Omega^2$$

r is defined by:

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$$

(T, R, θ, ϕ) - Kruskal coordinates

Note: $T = \pm R + \text{const}$

$\therefore$ the event horizon ($r = 2GM$) is defined by

$$T = \pm R$$

if $r = \text{const}$ $\quad T^2 - R^2 = \text{const}$

if $t = \text{const}$ $\rightarrow \frac{T}{R} = \tanh\left(\frac{t}{4GM}\right)$ – hyperbolae in R–T

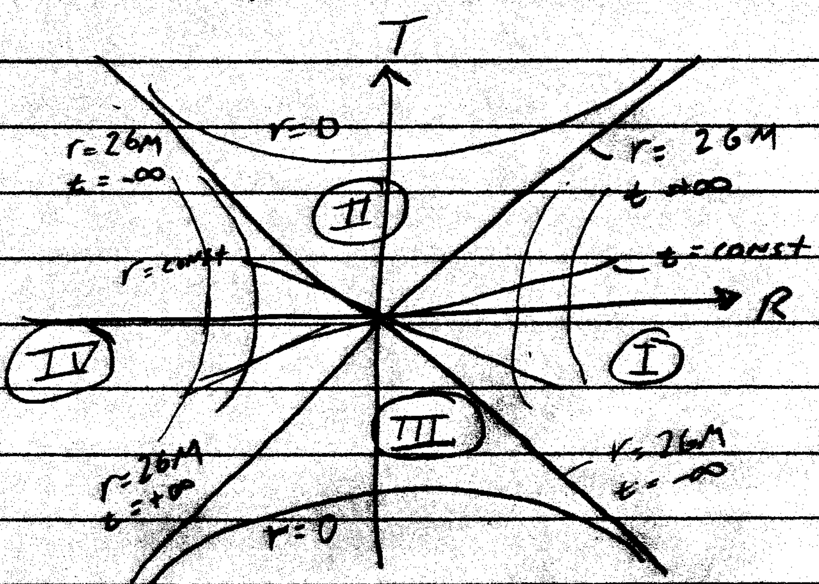
as $t \rightarrow \pm\infty$ $\quad \frac{T}{R} \rightarrow \pm 1$

$\therefore$ $t \rightarrow \pm\infty$ is also a the horizon $r = 2GM$

as long as $r \neq 0$ any values of T + R are allowed

$$-\infty \leq R \leq \infty$$
$$T^2 < R^2 + 1$$

Kruskal diagram
Fig 5.12



Original Schwarzschild is only good for $r > 2GM$

(I) ~ $r > 2GM$ (our universe)
(II) ~ future (black hole)
(III) ~ past
(IV) ~ related by space-like geodesics
(not reachable from (I))

All future directed paths go from region I to region II and end up inside the black hole and you are doomed

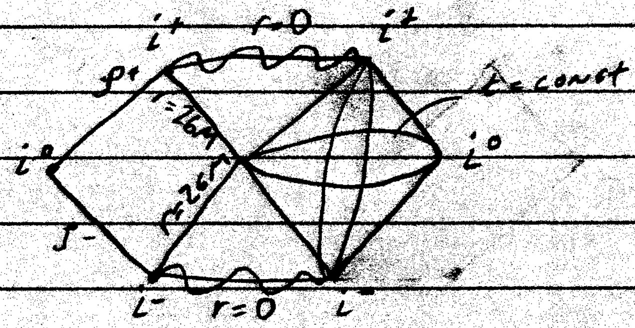
r becomes timelike + t becomes spacelike

BTW: tidal forces rip you apart and there is no excape

Region III is the time-reverse of region II called a white hole. There is a singularity in the past, nothing can enter but stuff can leave (past event horizon)

Region IV - parallel universe connected to Region I only by a wormhole (or Einstein-Rosen bridge)
(see Fig 5.15)
(Penrose)
Conformal diagrams (Appendix H) are also useful for studying spacetimes

Conformal diagram

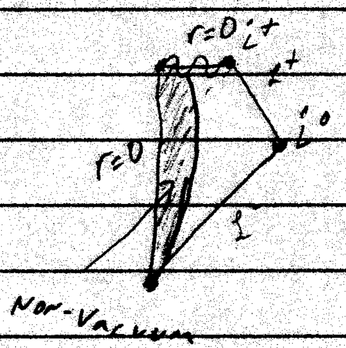


i^+ - future infinity
i^- - past infinity
i^0 - spacial infinity

$\mathscr{I}^+$ - future null infinity
$\mathscr{I}^-$ - past null infinity

Stars and Black Holes

conformal diagram for stellar collapse → Black Hole



To describe the process of stellar collapse we first look at spherically symmetric non-vacuum solutions to Einstein's Eqn.

This is similar to Schwarzschild except w/

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$$

This leads to the Schwarzschild mass

$$M = m(R) = 4\pi \int_0^R \rho(r) r^2 dr$$

radius of the star

true energy density

$$\bar{M} = \int_0^R \rho(r) \sqrt{|\gamma|} d^3x$$

$$= 4\pi \int_0^R \frac{\rho(r) r^2}{\left[1 - \frac{2Gm(r)}{r}\right]^{1/2}} dr$$

binding Energy

$$E_B = \bar{M} - M > 0$$

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]}$$

Hydrostatic Equilibrium Eqn
(Tolman-Oppenheimer-Volkoff eqn)

Know we have related $p(r)$ to $\rho(r)$

$\therefore$ $p(\rho)$ - equation of state

usually $p = K\rho^\gamma$

for a constant density star

$$\rho(r) = \begin{cases} \rho_* & r<R \\ 0 & r>R \end{cases}$$

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_* & r<R \\ \frac{4}{3}\pi R^3 \rho_* = M & r>R \end{cases}$$

$$\therefore\ p(r) = \rho_* \left[\frac{R\sqrt{R-2GM} - \sqrt{R^3-2GMr^2}}{\sqrt{R^3-2GMr^2} - 3R\sqrt{R-2GM}} \right]$$

if the mass of the star exceeds $M_{max} = \frac{4}{9G}R$ the central pressure $p(0)$ will be greater than infinite

$\therefore$ there are no static solutions to GR and the star collapses eventually forming a black hole

Buchdahl's Theorem - no static solutions exist unless $M < 4R/9G$

Chandrasekar limit - if an object has a mass above $1.4\, M_\odot$ it can collapse to a Neutron Star

Oppenheimer - Volkoff limit ~ 3-$4\, M_\odot$ collapse to a black hole

The Black Hole Zoo

For non-spherically symmetric cases there is an unlimited number of possible fields

no-hair theorem - Stationary, asymptotically flat black hole solutions to GR coupled to EM that are nonsingular outside the event horizon are fully characterized by mass, electric + magnetic charge and angular momentum.

If the no-hair theorem wasn't true, we would need an infinite number of multipoles to describe gravitational fields from non-spherically symmetric black holes

No matter how complex the initial conditions the final black hole is always the same