Geodesics of Schwarzschild

start w/ the NON-zero Christoffel symbols

$\Gamma^r_{tt} = \frac{GM}{r^3}(r-2GM)$ $\Gamma^r_{rr} = \frac{-GM}{r(r-2GM)}$ $\Gamma^t_{tr} = \frac{GM}{r(r-2GM)}$

$\Gamma^\theta_{r\theta} = \frac{1}{r}$ $\Gamma^r_{\theta\theta} = -(r-2GM)$ $\Gamma^\phi_{r\phi} = \frac{1}{r}$

$\Gamma^r_{\phi\phi} = -(r-2GM)\sin^2\theta$ $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ $\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

The geodesic eqns are then

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0$$

$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2$$

$$-(r-2GM)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

we use the 4-Killing vectors to simplify the problem

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}$$

$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ is also constant on the geodesic

for massive particles $\epsilon = -g_{\mu\nu} U^\mu U^\nu = +1$
for null trajectories $\epsilon = 0$

normalize λ so that for null geodesics $p^\mu = \frac{dx^\mu}{d\lambda}$

Note: time invariance $\Rightarrow$ cons of energy
spatial invariance $\rightarrow$ cons of momentum
rotational invariance $\rightarrow$ cons of ang momentum

cons of ang mom for a spherically symmetric space implies that we can use $\theta = \frac{\pi}{2}$ for a single particle

$$K^\mu = (\partial_t)^\mu = (1,0,0,0) \text{ - timelike Killing vector}$$

$$R^\mu = (\partial_\phi)^\mu = (0,0,0,1) \text{ - angular momentum Killing vector}$$

$$\Rightarrow K_\mu = \left(-\left(1 - \frac{2GM}{r}\right), 0, 0, 0\right)$$

(for geodesics of interest)

$$R_\mu = (0,0,0,r^2 \sin^2\theta) \Rightarrow (0,0,0,r^2)$$

$$\therefore E = -K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda}$$

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda}$$

for massless particles $E + L$ – energy + ang mom
for massive particles $E + L \sim \frac{\text{energy + ang mom}}{\text{mass}}$

$-p_\mu U^\mu$ - inertial/kinetic energy of the particle

$-p_\mu K^\mu$ - total conserved energy

the second const eqn implies

$$-\epsilon = -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

$$\Rightarrow -E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\epsilon\right) = 0$$

$$\Rightarrow \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r)$$

$$\mathcal{E} = \frac{1}{2}E^2 \,, \quad V(r) = \underset{\text{const}}{\frac{1}{2}\epsilon} - \underset{\text{Newton}}{\epsilon\frac{GM}{r}} + \underset{\text{ang mom}}{\frac{L^2}{2r^2}} - \underset{\text{GR}}{\frac{GML^2}{r^3}}$$

Newton vs Einstein
See p 210 + 211 in Carroll Fig 5.4 + 5.5

for $\frac{dV}{dr} = 0$ we get circular orbits $r_c = \text{const}$

$$\epsilon GMr_c^2 - L^2 r_c + 3GML^2\gamma = 0$$

$\gamma = 0$ - Newton
$\gamma = 1$ - Einstein

for Newton $r_c = \frac{L^2}{6GM}$

$E = 0$ for massless particles
∴ massless particles are not effected by gravity and can't move in a circle

massive bound particles move in circles or ellipses

unbound massive particles move in parabolas or hyperbolas

as $r \to \infty$ GR and Newton give almost the same answer

as $r \to 0$ Newton gives $V \to \infty$
Einstein gives $V \to 0$ @ $r = 2GM$

Massive: This implies stable orbits for Newton
" " unstable orbits for Einstein
if a particle loses enough energy

Massless: Newton - massless particles are never trapped
Einstein - massless particles are trapped as $r \to 0$

for GR w/ $\epsilon = 0$ - massless particle
$(\gamma = 1)$

$$r_c = 3GM$$ -- unstable radius for a circular orbit of a photon

for a massive particle $(\epsilon = 1)$

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}$$

2 radi: 1 stable, 1 unstable

as $L \to \infty$

$$r_c = \frac{L^2 \pm L^2(1 - 6G^2M^2/L^2)}{2GM} = \left(\frac{L^2}{GM},\ 3GM\right)$$

stable, same as Newton — unstable, same as photon

for $L = \sqrt{12}\,GM$ $\quad r_c = \frac{L^2}{2GM} = 6GM$ - 2 radi merge

for $L < \sqrt{12}\,GM$ no stable radi

$\therefore$ $6GM$ is the smallest possible stable circular orbit
InSCO - (Inner most Stable Circular Orbit)

Experimental Tests

Einstein suggested 3 tests: deflection of light
precession of perihelia *
gravitational redshift *

precession of perihelia - non-circular orbits in GR are not perfect closed ellipses

multiply $\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = E$ by $\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{r^4}{L^2}$

$$\Rightarrow \left(\frac{dr}{d\phi}\right)^2 + \frac{1}{L^2}r^4 - \frac{2GM}{L^2}r^3 + r^2 - 2GMr = \frac{2E}{L^2}r^4$$

$x = \frac{L^2}{GMr}$ $\therefore$ $x = 1$ for a Newtonial circular orbit

$$\Rightarrow \left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G^2M^2} - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 = \frac{2EL^2}{G^2M^2}$$

diff wrt ϕ

$$\Rightarrow \frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2}{L^2}x^2$$

if $x = x_0 + x_1$ - Newtonian + perturbation

$$\Rightarrow \frac{d^2x_0}{d\phi^2} - 1 + x_0 = 0 \quad , \quad \frac{d^2x_1}{d\phi^2} + x_1 \approx \frac{3G^2M^2}{L^2}x_0^2$$

$x_0 = 1 - e\cos\phi$ - from classical mechanics

$e^2 = 1 - b^2/a^2$

a - semi-major axis

b - semi-minor axis

$$\Rightarrow \frac{d^2x_1}{d\phi^2} + x_1 \approx \frac{3G^2M^2}{L^2}(1 + e\cos\phi)^2$$

$$\approx \frac{3G^2M^2}{L^2}\left[\left(1 + \frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right]$$

$$\frac{d^2}{d\phi^2}(\phi \sin\phi) + \phi \sin\phi = 2\cos\phi$$

and $$\frac{d^2}{d\phi^2}(\cos 2\phi) + \cos 2\phi = -3\cos 2\phi$$

$$\therefore \; x_1 \approx \frac{3G^2M^2}{L^2}\Big[\underbrace{(1+\tfrac{1}{2}e^2)}_{\text{const}} + \underbrace{e\phi\sin\phi}_{\text{grows w/ }\phi} - \underbrace{\tfrac{1}{6}e^2\cos 2\phi}_{\text{oscillates}}\Big]$$

$$\therefore \; x \approx 1 + e\cos\phi + \frac{3G^2M^2e}{L^2}\phi\sin\phi$$

$$\Rightarrow \; x \approx 1 + e\cos[(1-\alpha)\phi] \qquad \alpha = \frac{3G^2M^2}{L^2}$$

$$\cos[(1-\alpha)\phi] = \cos\phi + \alpha\frac{d}{d\alpha}\cos[(1-\alpha)\phi]_{\alpha=0}$$

$$= \cos\phi + \alpha\phi\sin\phi$$

$$\therefore \; \Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2}$$

An ordinary ellipse has

$$r = \frac{(1-e^2)a}{1+e\cos\phi} \quad \Rightarrow L^2 \approx GM(1-e^2)a$$

$$\therefore \; \Delta\phi = \frac{6\pi GM}{c^2(1-e^2)a}$$

1st test of GR - Mecury's orbit

$$\frac{GM_\odot}{c^2} = 1.48 \times 10^5 \text{ cm}$$

$a = 5.79 \times 10^{12}\,cm$

$e = 0.2056$

$\therefore\ \Delta\phi_{mercury} = 5.01 \times 10^{-7}\ \text{rads/orbit} = 0.103''/\text{orbit}$

$= 43.0''/\text{century}$

observed: $5601''/\text{century}$

$- 5025''/\text{century}$ - geocentric coord

$- 532''/\text{century}$ - other planets

$44''/\text{century}$ - GR

Redshift

timelike observer w/ $U^i = 0$ - stationary observer

$U_\mu U^\mu = -1 \quad U^0 = \left(1 - \frac{2GM}{r}\right)^{-1/2}$

$\omega = -g_{\mu\nu} U^\mu \frac{dx^\nu}{d\lambda}$ - freq of a photon on a null geodesic

$\therefore\ \omega = \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{dt}{d\lambda} = \left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} E$

$\Rightarrow\ \frac{\omega_2}{\omega_1} = \left(\frac{1 - 2GM/r_1}{1 - 2GM/r_2}\right)^{1/2}$

for $r \gg 2GM$

Newtonian potential

$$\frac{\omega_2}{\omega_1} \approx 1 - \frac{GM}{r_1} + \frac{GM}{r_2} = 1 + \Phi_1 - \Phi_2$$

freq decreases as Φ increases

1st detected in 1960 by Pound and Rebka