The Schwarzschild Solution

To approximate the gravitational field outside the Earth or Sun we need a spherically symmetric gravitational field in empty space

Schwarzschild metic - unique spherically symmetric vacuum solution to Einstein's Eqn.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

M - mass of gravitating object

in a vacuum $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$

the source is static + spherically symmetric

static - no time-space cross terms $g_{0i} = g_{i0} = 0$,
- time-symmetric $g_{\mu\nu}(t) = g_{\mu\nu}(-t)$

spherically symmetric Minkowski space

$$ds^2_{Minkowski} = -dt^2 + dr^2 + r^2 d\Omega^2$$

$\Rightarrow \quad ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega^2$

same signature and still spherically symmetric

change coords - $\bar{r} = e^{\gamma(r)} r$

$$\Rightarrow d\bar{r} = e^{\gamma}dr + e^{\gamma} r \, d\gamma = \left(1 + r\frac{d\gamma}{dr}\right)e^{\gamma}dr$$

$$\therefore ds^2 = -e^{2\alpha(r)}dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r) - 2\gamma(r)} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

if $\bar{r} \to r$, $\left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta - 2\gamma} \to e^{2\beta}$

$$\therefore ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr + r^2 d\Omega^2$$

solve for Christoffel symbols

$\Gamma^t_{tr} = \partial_r \alpha \qquad \Gamma^r_{tt} = e^{2(\alpha - \beta)}\partial_r \alpha \qquad \Gamma^r_{rr} = \partial_r \beta$

$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -re^{-2\beta} \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$

$\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

all others are zero or symmetrically related

$$R^{t}{}_{rtr} = \partial_r \alpha \partial_r \beta - \partial_r^2 \alpha - (\partial_r \alpha)^2$$

$$R^{t}{}_{\theta t \theta} = -r e^{-2\beta} \partial_r \alpha$$

$$R^{t}{}_{\phi t \phi} = -r e^{-2\beta} \sin^2\theta \, \partial_r \alpha$$

$$R^{r}{}_{\theta r \theta} = r e^{-2\beta} \partial_r \beta$$

$$R^{r}{}_{\phi r \phi} = r e^{-2\beta} \sin^2\theta \, \partial_r \beta$$

$$R^{\theta}{}_{\phi \theta \phi} = (1 - e^{-2\beta}) \sin^2\theta$$

contract to the Ricci tensor

$$R_{tt} = e^{2(\alpha-\beta)} \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_R \beta + \frac{2}{r} \partial_r \alpha \right]$$

$$R_{rr} = - \partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta$$

$$R_{\theta\theta} = e^{-2\beta} [r(\partial_r \beta - \partial_r \alpha) - 1] + 1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

contract again for the Ricci Scalar

$$R = -2 e^{-2\beta} \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r}(\partial_r \alpha - \partial_r \beta) + \frac{1}{r^2}(1 - e^{2\beta}) \right.$$

$R_{\rho r} = 0$. $R_{tt} = 0$, $R_{rr} = 0$ $\therefore$ $kR_{tt} + R_{rr} = 0$

$\therefore \quad 0 = e^{2(\beta - \alpha)} R_{tt} + R_{rr} = \frac{2}{r}(\partial_r \alpha + \partial_r \beta)$

$\Rightarrow \alpha = -\beta + c$

if $t \to e^{-c} t \qquad \alpha = -\beta$

$R_{\theta\theta} = 0$

$\therefore \quad 1 = -e^{-2\beta}[r(\partial_r \beta - \partial_r \alpha) - 1]$

$\Rightarrow 1 = e^{+2\alpha}[2r\partial_r \alpha + 1]$

$\Rightarrow \partial_r(re^{2\alpha}) = 1 \quad \to \quad re^{2\alpha} = r - R_s$

$\therefore e^{2\alpha} = 1 - \frac{R_s}{r}$

$\therefore \quad ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$

R_s - Schwarzschild radius

for $r \gg 2GM$ ~~this~~ is the weak-field limit

we only need to identify R_s with $2GM$

asymptotically flat - Minkowski space as $r \to \infty$

Birkhoff's Theorem

- The Schwarzschild metric is the unique vacuum solution w/ spherical symmetry

- There are no time-dependent solutions of this form

Proof

1. argue that a spherically symmetric space-time can be foliated by 2-spheres
2. show geometrically that the metric on a spherically symmetric space can be written as:

$$ds^2 = d\tau^2(a,b) + r^2(a,b)\, d\Omega^2(\theta,\phi)$$

transverse to sphere

3. plug metric into Einstein's eqn for a vacuum to show that Schwarzschild's solution is unique

1. spherically symmetric $\Rightarrow$ $SO(3)$ symmetry

3 Killing vectors are for this space

$$R = \partial_\phi$$
$$S = \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi$$
$$T = -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\phi$$

$$[R,S] = T \quad [S,T] = R \quad [T,R] = S$$

true if a manifold is spherically symmetric

Since the vector fields that obey the commutation relations are 2-spheres, every point will be on exactly one of the spheres

$\therefore$ the spherical manifold can be foliated into spheres

2. (θ, ϕ) - coords on each sphere
(a, b) - coords of each sphere in the set
(a, b, θ, ϕ) - complete set of coords on M

for each sphere $a = a_0$, $b = b_0$

$\therefore$ for a chosen sphere

$$ds^2(a_0, b_0, \theta, \phi) = f(a_0, b_0)\, d\Omega^2$$

ind of $\theta + \phi$ and therefore round

for fixed $\theta = \theta_0$ and $\phi = \phi_0$

$$ds^2(a, b, \theta_0, \phi_0) = d\tau^2(a, b)$$

$$\therefore\ ds^2(a, b, \theta, \phi) = d\tau^2(a, b) + f(a, b)\, d\Omega^2$$

using several coord changes Carroll p200-201

$$ds^2 = m(t, r)\, dt^2 + n(t, r)\, dr^2 + r^2 d\Omega^2$$

3)

$\Rightarrow ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2$

same as before except α + β are functions of r + t

$\Rightarrow R_{tt} = [\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\,\partial_t\beta]$
$+ e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\alpha]$

$R_{rr} = -[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta - \frac{2}{r}\partial_r\beta]$
$+ e^{2(\beta-\alpha)}[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\,\partial_t\beta]$

$R_{tr} = \frac{2}{r}\partial_t\beta$

$R_{\theta\theta} = e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1$

$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$

$R_{\mu\nu} = 0 \quad \therefore \; R_{tr} = 0 \Rightarrow \partial_t\beta = 0$

$\beta(t,r) \rightarrow \beta(r)$

$R_{\theta\theta} = 0 \Rightarrow \partial_t R_{\theta\theta} \Rightarrow \partial_t\partial_r\alpha = 0$

$\therefore \; \alpha(r,t) \rightarrow f(r) + g(t)$

if we choose $dt \rightarrow e^{-g(t)}dt \quad \alpha(t,r) \rightarrow f(r)$

$$\Rightarrow \quad ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

metric is ind of t

any spherically symmetric vacuum metric has a timelike killing vector proving Birkhoff's theorem

Stationary - metric possesses a Killing vector that is timelike near infinity

Static - metric possesses a timelike Killing vector that is orthogonal to a family of hypersurfaces

general form $\Rightarrow ds^2 = g_{00}(\vec{x})dt^2 + g_{ij}(\vec{x})dx^i dx^j$

time-symmetric

Singularities

metric coefficients become infinite at $r=0$ and $r=2GM$ (R_s)

This implys that something ~~is~~ wrong

Curvature is infinite at these pts because it depends on the coords but the scalars do not so if none of them are infinite there is no singularity these scalars include:

$$R, \quad R^{\mu\nu}R_{\mu\nu}, \quad R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \quad R_{\mu\nu\rho\sigma}R^{\rho\sigma\lambda\tau}R_{\lambda\tau}{}^{\mu\nu}$$

we will consider a pt non-singular if it has well-behaved geodesics

for Schwarzschild $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$

at $r=0$ there is a singularity

at $r=2GM$ there isn't

$r=2GM$ is called a coordinate singularity changing the coords removes it

$r=2GM$ is the event horizon of a black hole

for our sun this is inside the star where our assumptions about a vacuum are invalid