Properties

Einstein's Eqn is actually 10 different eqns because of the rank 2 symmetric tensors which it is composed of (in 4 Dimensions)

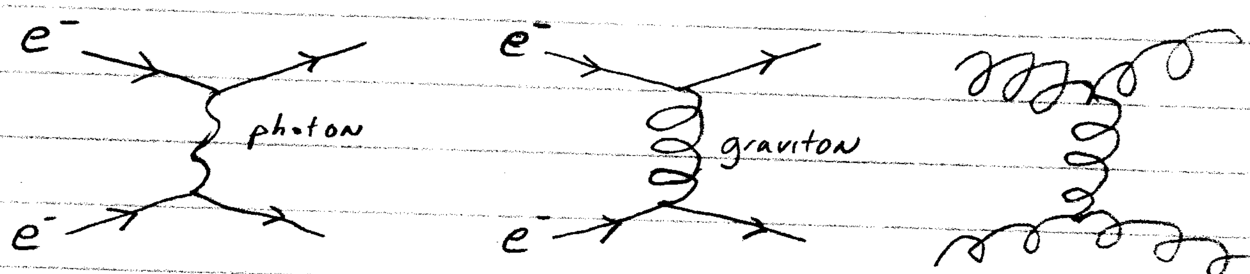
the Banchi identity $\nabla^{\mu} G_{\mu\nu} = 0$ represents 4 constraints on $R_{\mu\nu}$ $\therefore$ there are 6 ind eqns (from translations)

These are difficult to solve and non-linear for large curvature since they involve products and derivatives of the metric.

Symmetry and assuming a vacume simplify the process of finding solution

Nonlinearitys can be physically interpreted as the gravitational field coupling to itself

Feynman diagrams - used in QFT to calculate scattering



electron EM interaction

gravitational interaction

gravitons interacting w/ gravitons

EM is actually the only linear gauge theory since it is U(1) abelian (commutes)

Gravity can be described by a spin 2 graviton and the SO(3,1) non abelian group

From ball example; please read 167-169

$$\frac{d\theta}{d\tau} = 2\omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2 - R_{\mu\nu}U^\mu U^\nu$$ ← Raychaudhuri's eqn

where U^ν - four velocities of particles moving on geodesics

$\theta = \nabla_\mu U^\mu$ - expansion

σ - shear

ω - rotation

$$\Rightarrow \frac{d\theta}{d\tau} = -4\pi G(\rho + p_x + p_y + p_z)$$ using Einstein's Eqn

for $\rho > 0 = p_x + p_y + p_z$ the system contracts

$\therefore$ gravity is attractive for positive mass density

The expansion of the volume of any set of particles initially at rest is proportional to (minus) the sum of the energy density and the three components of pressure

What about the Weyl tensor?

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \frac{1}{3} g_{\rho[\mu} g_{\nu]\sigma} R - g_{\rho[\mu} R_{\nu]\sigma} + g_{\sigma[\mu} R_{\nu]\rho}$$

trace-free part of Riemann

Given a specified energy-momentum distribution, there is still some freedom in the choice of Weyl since there is no eqn relating $C^{\rho}{}_{\sigma\mu\nu}$ to $T_{\mu\nu}$

given the Bianchi identity $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\nabla^{\rho} C_{\rho\sigma\mu\nu} = \nabla_{[\mu} R_{\nu]\sigma} + \frac{1}{6} g_{\sigma[\mu} \nabla_{\nu]} R$$

$$= 8\pi G \left(\nabla_{[\mu} T_{\nu]\sigma} + \frac{1}{3} g_{\sigma[\mu} \nabla_{\nu]} T \right)$$

1^{st} order so solutions will depend on boundary conditions "propagation" eqn for gravitational waves similar to $\nabla_{\mu} F^{\nu\mu} = J^{\nu}$

Einstein's GR works above the quantum scale

$\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-27} \frac{cm^2 g}{s}$

$G = 6.67 \times 10^{-8} cm^3/g s^2$

$c = 3.00 \times 10^{10} cm/s$

$m_p = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-5} g$

$l_p = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-33} cm$

$t_p = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44} s$

$E_p = \left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.95 \times 10^{16} erg$

$1.22 \times 10^{19} GeV$

The Cosmological Constant

Because the source of gravity in GR is the energy-momentum tensor (instead of differences in energy like in Newtonian Physics) the normalization of energy is not arbitrary.

Vacuum energy - energy density of empty space

For space to be isotropic the vacuum energy-momentum tensor should be Lorentz invariant in locally inertial coords

$$\therefore \quad T^{(vac)}_{\hat{\mu}\hat{\nu}} = -\rho_{vac}\,\eta_{\hat{\mu}\hat{\nu}}$$

$$\Rightarrow T^{(vac)}_{\mu\nu} = -\rho_{vac}\, g_{\mu\nu}$$

$$\therefore\ p_{vac} = -\rho_{vac} \Leftarrow T_{\mu\nu} = (\rho + p)\,U_\mu U_\nu + p\, g_{\mu\nu}$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G\big(\underbrace{T^{(M)}_{\mu\nu}}_{\text{matter}} - \underbrace{\rho_{vac}\, g_{\mu\nu}}_{\text{vacuum}}\big)$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\rho_{vac} = \frac{\Lambda}{8\pi G}$$ $\Lambda \leftarrow$ cosmological constant

using the cc in the action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} (R - 2\Lambda) + \hat{\mathcal{L}}_m \right]$$

$$\Rightarrow \hat{\mathcal{L}}_{vac} = -\rho_{vac}$$

constants also satisfy least action wrt the metric

Among the sources of vacuum energy are zero-point quantum fluctuations

In QM the lowest allowed energy state is $E_0 = \frac{1}{2}\hbar\omega$. For each allowed frequency this results in an infinite energy density.

for a cutoff wave number ~~k_{max}~~ (ultraviolet momentum cutoff)

$$\rho_{vac} \sim \hbar k_{max}^4$$

assuming that QFT works all the way up to the Planck scale

$$\bar{m}_P = (8\pi G)^{-1/2} \sim 10^{18}\, GeV$$

$$\therefore \quad \rho_{vac} \sim (10^{18}\, GeV)^4 \sim 10^{112}\, erg/cm^3$$

cosmological observations imply

$$|\rho_\Lambda^{(obs)}| \leq (10^{-12}\, GeV)^4 \sim 10^{-8}\, erg/cm^3$$

off by 120 orders of magnetude

We don't know why the cutoff should be so much higher than the Planck scale "cosmological constant problem"

Energy conditions

goal: constraint $T_{\mu\nu}$ for physically allowable metrics using "realistic" energy-momentum sources

construct scalars from $T_{\mu\nu}$ using timelike t^μ and null-like ℓ^μ vectors

Example:

given $T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$

$$T_{\mu\nu} t^\mu t^\nu = (\rho + p) U_\mu t^\mu U_\nu t^\nu + p g_{\mu\nu} t^\mu t^\nu$$

$$= (\rho + p) K^2 - p(t)^2 \quad \text{where } K^2 \geq t^2$$

if U_μ is timelike

$\therefore T_{\mu\nu} t^\mu t^\nu \geq 0$ if $\rho \geq 0$ and $\rho + p \geq 0$

Weak Energy Condition - (WEC)

$T_{\mu\nu} t^\mu t^\nu \geq 0$ for all t^μ $\therefore$ $\rho \geq 0$, $\rho + p \geq 0$

timelike

pressure is small compared to energy density

Null Energy Condition - (NEC)

energy may be negative

$T_{\mu\nu} l^\mu l^\nu \geq 0$ for all l^μ $\therefore$ $\rho + p \geq 0$

null vectors

Dominate Energy Condition - (DEC)

$T_{\mu\nu} t^\mu t^\nu \geq 0$ for all t^μ .

energy is not negative

$\therefore$ $\rho \geq |p|$

$T^{\mu\nu} t_\mu$ is not spacelike

greater than pressure

Null Dominant Energy Condition - (NDEC)

$T_{\mu\nu} l^\mu l^\nu \geq 0$ for all l^μ

neg energy is allowed

$\therefore$ $p = -\rho$

$T^{\mu\nu} l_\mu$ is not spacelike

Strong Energy Condition (SEC)

No large neg pressure

$T_{\mu\nu} t^\mu t^\nu \geq \frac{1}{2} T^\lambda{}_\lambda t^\sigma t_\sigma$ for all t^μ

implies gravity is attractive $\therefore$ $\rho + p \geq 0$, $\rho + 3p \geq 0$

eqn of state parameter $w = P/\rho$

see fig 4.3 p 176

Most classical matter obeys DEC
and therefore WEC, NEC + NDEC

SEC may be violated by massive scalar fields

Quantum Fields may violate the energy conditions shown

Equivalence Principle Revisited

used to justify

1) Physical Laws should be expressible in a covariant form

2) There is a metric on spacetime and gravity
is it's curvature

3) There are no other fields like gravity

4) There is a minimal amount of interaction
between matter fields and curvature

statement 1) all the terms in the eqn transform the same way under a change of coords i.e. the form of the eqn is coord-invariant

We can express "Physical laws in terms of tensors making them coord independent

$$\text{Ex.} \quad \partial_\mu F^{\nu\mu} = J^\nu \rightarrow \nabla_\mu F^{\nu\mu} = J^\nu$$

statement 2) EP implies gravity is universal and therefore impossible to measure in small regions of spacetime i.e. gravity can be linked to geometry

statement 3) tests of the EP also test for other fields that resemble gravity (long-range and couples almost universally to mass). Detecting a violation could imply a 5th force

statement 4) couplings are minimal

what if
$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = \alpha (\nabla_\sigma R) \frac{dx^\mu}{d\lambda} \frac{dx^\sigma}{d\lambda}$$

it would alow for direct detection of curvature in small regions using $\nabla_\sigma R$

from dimensional analysis $[\alpha] = L^2$

if $\alpha \sim l_p^2$ (the natural scale),

this implies a RHS $\sim 10^{-102}$ times the LHS for the gravitational field of the Earth this is not currently detectible

∴ The Principle of Equivalence is a consequence of GR being a macroscopic effective field theory.

Alternative Theories

some modifications of GR

- gravitational scalar fields
- extra (spatial) dimensions
- higher order action terms
- non-Christoffel connections

Scalar-tensor theories - use both a metric $g_{\mu\nu}$ and a scalar field λ which couples to the curvature scalar the action is

$$S = S_{fR} + S_\lambda + S_m$$

curvature $\rightarrow$ $$S_{fR} = \int d^4x \sqrt{-g}\, f(\lambda) R$$

scalar $\rightarrow$ $$S_\lambda = \int d^4x \sqrt{-g} \left[-\frac{1}{2} h(\lambda) g^{\mu\nu} (\partial_\mu \lambda)(\partial_\nu \lambda) - U(\lambda) \right]$$

matter $\rightarrow$ $$S_m = \int d^4x \sqrt{-g}\, \hat{\mathcal{L}}_m (g_{\mu\nu}, \Psi_i)$$

$f(\lambda)$, $h(\lambda)$ + $U(\lambda)$ are functions

Ψ_i - matter fields

δS comes from $\delta g^{\mu\nu}$ and $\delta\lambda$

$$\delta S_{fR} = \int d^4x \sqrt{-g} \left[f(\lambda) G_{\mu\nu} + g_{\mu\nu} \Box f - \nabla_\mu \nabla_\nu f \right] \delta g^{\mu\nu}$$

$$G_{\mu\nu} = f^{-1}(\lambda) \left(\frac{1}{2} T_{\mu\nu}^{(m)} + \frac{1}{2} T_{\mu\nu}^{(\lambda)} + \nabla_\mu \nabla_\nu f - g_{\mu\nu} \Box f \right)$$

$$T_{\mu\nu}^{(\lambda)} = h(\lambda) \nabla_\mu \lambda \nabla_\nu \lambda - g_{\mu\nu} \left[\frac{1}{2} h(\lambda) g^{\rho\sigma} \nabla_\rho \lambda \nabla_\sigma \lambda + U(\lambda) \right]$$

if λ varies with position it implies a space-time dependent Newton's constant

λ obeys the eqn

$$h \Box \lambda + \frac{1}{2} h' g^{\mu\nu} \nabla_\mu \lambda \nabla_\nu \lambda - U' + f' R = 0$$

Brans - Dicke theory

$f(\lambda) = \frac{\lambda}{16\pi}$ $h(\lambda) = \frac{\omega}{8\pi\lambda}$ $U(\lambda) = 0$

$$S_{BD} = \int d^4x \sqrt{-g} \left[\frac{\lambda}{16\pi} R - \frac{\omega}{16\pi} g^{\mu\nu} \frac{(\partial_\mu \lambda)(\partial_\nu \lambda)}{\lambda} \right]$$

$\omega \to \infty$ implies ordinary GR

for our solar system $\omega > 500$

Extra Dimensions

$4+d$ total dimensions but d-dimensions are compactified to appear 4 dimensional at large distances

These are known as Kaluza - Klein Theories