Physics on Curved Spacetime

2 parts of any theory of gravity

1. how gravity influences matter

2. how matter affects gravity

for Newton:

$$a = -\nabla \Phi$$

$$\nabla^2 \Phi = 4\pi G \rho$$

for GR we use curvature

curvature acts on matter

matter causes curvature

using EEP - it is impossible to detect a gravitational field using local experiments because gravity is universal in its effects ~~on~~ all particles and energy.

Because it affects everything we can think of gravitation as a result of curvature

minimal-coupling principle

1) Start w/ physics in inertial coords of flat space

2) Write in coord-invariant form

3) Assert the law is true in general coords

$\eta_{\mu\nu} \rightarrow g_{\mu\nu} \qquad \partial_\mu \rightarrow \nabla_\mu$

in flat space + inertial coords

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \qquad x^\mu(\lambda) \text{ - path}$$

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{dx^\nu}{d\lambda}\partial_\nu \frac{dx^\mu}{d\lambda} \longrightarrow \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda}$$

$$\Rightarrow \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}$$

$$\therefore \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$ for an unaccelerated particle in general coordinates

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energy-momentum conservation $\qquad \partial_\mu T^{\mu\nu} = 0 \Rightarrow \nabla_\mu T^{\mu\nu} = 0$

The Newtonian Limit

1) particles move much slower than light
2) gravity is weak
3) the gravitational field is static

$$1) \Rightarrow \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \qquad \therefore \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

ignore $\frac{dx^i}{d\tau}$

3) for a static field $\partial_0 g_{\mu\nu} = 0$ - metric is const in time

$$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\mu\lambda}(\cancel{\partial_0 g_{\lambda 0}}^{0} + \cancel{\partial_0 g_{0\lambda}}^{0} - \partial_\lambda g_{00})$$

$$= -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00}$$

2) $\Rightarrow$ for weak gravity $g_{\mu\nu} = \eta_{\mu\nu} + \underset{\substack{\uparrow \\ \text{perturbation}}}{h_{\mu\nu}}$ $|h_{\mu\nu}| \ll 1$

$$\therefore g^{\mu\nu} g_{\mu\nu} = \eta^{\mu\nu}\eta_{\mu\nu} + \eta^{\mu\nu} h_{\mu\nu} - \eta_{\mu\nu} h^{\mu\nu} + h^{\mu\nu} h_{\mu\nu} \Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$\Rightarrow h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}$$

$$\therefore \Gamma^{\mu}_{00} \approx -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}$$

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2$$

for $\mu = 0$ $\quad \frac{d^2x^0}{d\tau^2} = 0$ since $\partial_0 h_{00} = 0$

for $\mu = i$ $\quad \frac{d^2x^i}{d\tau^2} = \frac{1}{2}\partial_i h_{00}\left(\frac{dt}{d\tau}\right)^2$

$$\Rightarrow \frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}$$

$\underset{\substack{\uparrow \\ \text{for a point}}}{\Phi} = -\frac{GM}{r}$ if $h_{00} = -2\Phi \Rightarrow g_{00} = -(1+2\Phi)$

$$\frac{d^2x^i}{dt^2} = -\partial_i \Phi \quad \text{- Newton's Gravity}$$

$$a = -\nabla\Phi$$

Einstein's Eqn

use the Poisson eqn for the Newtonian potential as an model

$$\nabla^2 \Phi = 4\pi G \rho$$

2nd order derivative — Mass distribution

1st guess

$$[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}$$

metric replaces potential — generalization of mass-energy distribution

LHS is not a tensor since $\nabla^2 g_{\mu\nu} = \nabla_\sigma \nabla^\sigma g_{\mu\nu} = 0$

however $R^\rho{}_{\sigma\mu\nu}$ is not zero and constructed from second derivatives

2nd guess

$$R_{\mu\nu} = \kappa T_{\mu\nu}$$

contracted for consistency

but $\nabla^\mu T_{\mu\nu} = 0$ $\quad \nabla^\mu R_{\mu\nu} \stackrel{?}{=} 0$ ← not always true from the Bianchi identity

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$$

$$\therefore \nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\nabla^\mu G_{\mu\nu} = 0 = \kappa \nabla^\mu T_{\mu\nu}$$

$$g^{\mu\nu}G_{\mu\nu} = R - \frac{1}{2}\delta^\mu_\mu R = \kappa g^{\mu\nu}T_{\mu\nu} = \kappa T$$

in 4D $\Rightarrow R = -\kappa T$

$$\therefore R_{\mu\nu} = \kappa T_{\mu\nu} - \frac{1}{2}\kappa g_{\mu\nu}T = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

if $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$

for dust $T_{\mu\nu} = \rho U_\mu U_\nu$

at rest $U^\mu = (U^0, 0, 0, 0)$

$$g_{\mu\nu}U^\mu U^\nu = -1$$

$$g_{00} = -1 + h_{00}$$
$$g^{00} = -1 - h^{00}$$

$$\Rightarrow g_{00} U^0 U^0 = -1$$

$$(-1 + h_{00})(U^0)^2 = -1$$

$$\therefore U^0 = \sqrt{\frac{-1}{-1+h_{00}}} = (1-h_{00})^{-\frac{1}{2}} \approx 1 + \frac{1}{2} h_{00}$$

$$U^0 \approx 1 \quad U_0 \approx -1 \quad \therefore T_{00} = \rho$$

$$T = g^{00} T_{00} = -T_{00} = -\rho$$

$$\therefore R_{00} = k\left(\rho - \frac{1}{2}\rho\right) = \frac{1}{2} k \rho$$

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = R^{i}{}_{0i0}$$

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \overset{0}{\cancel{\partial_0 \Gamma^{i}_{j0}}} + \overset{|h|^2}{\Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00}} - \overset{|h|^2}{\Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0}}$$

$$\therefore R^{i}{}_{0j0} \approx \partial_j \Gamma^{i}_{00}$$

$$R_{00} = R^{i}{}_{0i0} = \partial_i\left(\frac{1}{2} g^{i\lambda}\left(\overset{0}{\cancel{\partial_0 g_{\lambda 0}}} + \overset{0}{\cancel{\partial_0 g_{0\lambda}}} - \partial_\lambda g_{00}\right)\right)$$

$$= -\frac{1}{2}\left[\overset{0}{\cancel{\partial_i g^{i\lambda}}}(\partial_\lambda g_{00}) + g^{i\lambda}\partial_i \partial_\lambda g_{00}\right]$$

$$= -\frac{1}{2}\partial_i \partial^i g_{00}$$

$$= -\frac{1}{2}\nabla^2 h_{00}$$

$\therefore \nabla^2 h_{00} = -k\rho$

since $h_{00} = -2\Phi$

$k = 8\pi G$

so $\nabla^2 \Phi = 4\pi G\rho$

$\therefore R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$

or $R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu})$

Lagrangian Formulation

$S = \int \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) d^n x$ — generalized action in n-dimensions

$\underset{\text{density}}{\mathcal{L}} = \sqrt{|g|}\, \underset{\text{scalar}}{\hat{\mathcal{L}}}$

$\delta S = 0 \Rightarrow \frac{\partial \hat{\mathcal{L}}}{\partial \Phi} - \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial (\nabla_\mu \Phi)} \right) = 0$

use Stoke's Theorem

$$\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|}\, d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|}\, d^{n-1} x$$

and IBP's

$$\int A^\mu (\nabla_\mu B) \sqrt{|g|}\, d^n x = -\int (\nabla_\mu A^\mu) B \sqrt{|g|}\, d^n x + \text{boundary terms}$$

$\therefore\ S_\phi = \int \left[-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi)\right] \sqrt{|g|}\, d^n x$

$\mathcal{L}$ for a scalar field

$$\Rightarrow \quad \Box \phi - \frac{dV}{d\phi} = 0$$

where $\Box = \nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu$.

for GR $g_{\mu\nu}$ is the dynamic variable, R is the Lagrangian
R is the only independent scalar that can be constructed from the metric using no higher than 2nd order derivatives

$$S_H = \int R \sqrt{|g|}\, d^n x \quad \text{- Hilbert action}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\therefore\ \delta S_H = (\delta S)_1 + (\delta S)_2 + (\delta S)_3$$

$$(\delta S)_1 = \int d^n x \sqrt{|g|}\, g^{\mu\nu} \delta R_{\mu\nu}$$

$$(\delta S)_2 = \int d^n x \sqrt{|g|}\, R_{\mu\nu} \delta g^{\mu\nu}$$

$$(\delta S)_3 = \int d^n x\, R\, \delta \sqrt{|g|}$$

to find $\delta R_{\mu\nu}$ use

$$R^\rho{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu)$$

$$\Gamma^{\rho}_{\nu\mu} \to \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu}$$

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu})$$

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}$$

$$\therefore (\delta S)_1 = \int d^n x \sqrt{|g|}\, g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^{\lambda}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\right]$$

$$= \int d^n x \sqrt{|g|}\, \nabla_\sigma\left[g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\lambda\mu})\right]$$

$$\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2}\left[g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\right]$$

$$\therefore (\delta S)_1 = \int d^n x \sqrt{|g|}\, \nabla_\sigma\left[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right]$$

$$= \int_\Sigma d^{n-1} \sqrt{|\gamma|}\, n_\sigma\left[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right]$$

$= 0$ if the variation vanishes at the boundary of infinity

For $(\delta S)_3$

use $\ln(\det M) = \mathrm{Tr}(\ln M)$

$M = \exp(\ln M)$ for any matrix M

$$\therefore \frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M)$$

$$\therefore \frac{1}{g}\delta g = g^{\mu\nu}\delta g_{\mu\nu} \Rightarrow \delta g = -g(g_{\mu\nu}\delta g^{\mu\nu})$$

since $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$

$$\delta\sqrt{|g|} = -\frac{1}{2\sqrt{|g|}}\,\delta g$$

$$= \frac{1}{2}\,\frac{g}{\sqrt{|g|}}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\sqrt{|g|}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\therefore\ (\delta S)_3 = -\frac{1}{2}\int d^n x\, R\sqrt{|g|}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\Rightarrow\ \delta S_H = \int d^n x\sqrt{|g|}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu}$$

since $\delta S = \int \sum_i \left(\frac{\delta S}{\delta \Phi^i}\,\delta\Phi^i\right) d^n x$

$$\frac{1}{\sqrt{|g|}}\,\frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 0$$

$\delta S = 0$ in a vacume for least action

matter/energy fields add constraints

if $S = \frac{1}{16\pi G}S_H + S_m$

$$\frac{1}{\sqrt{|g|}}\,\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}\right) + \frac{1}{\sqrt{|g|}}\,\frac{\delta S_m}{\delta g^{\mu\nu}} = 0$$

$$T_{\mu\nu} \equiv -2\,\frac{1}{\sqrt{|g|}}\,\frac{\delta S_m}{\delta g^{\mu\nu}}$$

$$\therefore\ R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 8\pi G\,T_{\mu\nu}$$

Remember for a scalar field varying w/ the metric

$$\delta S_\phi = \int d^n x \left[\sqrt{|g|} \left(-\tfrac{1}{2} \delta g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) \right.$$

$$\left. + \delta \sqrt{|g|} \left(-\tfrac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \right]$$

$$= \int d^n x \sqrt{|g|}\, \delta g^{\mu\nu} \left[-\tfrac{1}{2} \nabla_\mu \phi \nabla_\nu \phi + \right.$$

$$\left. \left(-\tfrac{1}{2} g_{\mu\nu}\right)\left(-\tfrac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - V(\phi) \right) \right]$$

$$\therefore \quad T^{(\phi)}_{\mu\nu} = -2 \frac{1}{\sqrt{|g|}} \frac{\delta S_\phi}{\delta g^{\mu\nu}}$$

$$= \nabla_\mu \phi \nabla_\nu \phi - \tfrac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi)$$

in flat space

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} \eta_{\mu\nu} \partial^\sigma \phi \partial_\sigma \phi - \eta_{\mu\nu} V(\phi)$$

Noether's Theorem – every symmetry of a Lagrangian impiles the existence of a conservation Law

$\partial_\mu S^{\mu\nu} = 0$ where $S^{\mu\nu}$ is given by the variation of the Lagrangian

for a scalar field in Minkowski spacetime

canonical EM tensor $\rightarrow$ $S^{\mu\nu} = \dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^i)} \partial^\nu \Phi^i - \eta^{\mu\nu} \mathcal{L}$ – similar to the Hamiltonian density