Special Relativity + Flat Spacetime (1)

Prelude – Essential idea of GR:

- most forces are defined by fields on spacetime
- gravity is inherent in spacetime itself
- gravity is a manifestation of the curvature of spacetime

∴ We need to understand spacetime
" " " " how curvature becomes gravity

chap 1 + 2 are about spacetime
chap 3 is about curvature
chap 4 links curvature to gravity

2 basic elements of a gravity theory

1. eq for gravitational field as influenced by matter
2. eq for the response of matter to the field

Newton's Theory: $\vec{F} = \frac{-GMm}{r^2}\hat{r}$ ← inverse-square law

$\vec{F} = -m\nabla\Phi$ $\qquad \vec{F} = m\vec{a}$ ← Newton's 2nd law

$\nabla^2\Phi = 4\pi G\rho$ ← Poisson's eqn

$\vec{a} = -\nabla\Phi$ ← potential

for GR we need to replace these eqns with ones envolving the curvature of spacetime

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{\text{curvature of spacetime}} = \underbrace{8\pi G T_{\mu\nu}}_{\text{Mass/Energy}}$$ ← relates energy to curvature

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$ ← movement of mass through curved spacetime

geodesic eqn

In the relativistic view, gravity is not actually a "force", It is a feature of spacetime

a particle moving along a geodesic path $x^\mu(\lambda)$ does not feel acceleration like a charged particle in a B-field

Example: ball falling vs one sitting on a table

metric tensor ($g_{\mu\nu}$) - describes spacetime curvature

With curvature our idea of geometry must be modifie the metric in flat space represents (in cart coords)

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

In curved space Pythagoras' relation is altered
the metric contains information on the alteration

$g_{\mu\nu} \rightarrow R_{\mu\nu\sigma\rho} \rightarrow R_{\mu\nu} \rightarrow$ Einstein's eqns (LHS)
+ Geodesic Eqn

The real difficulty in GR is accepting the
concept of spacetime curvature although
it is not observed in daily life

Space + Time - Manifold - mathematical structure used
to describe spacetime

Einstein
vs
Newton

Special Relativity - model of 1 kind of spacetime

We will re-introduce special relativity as a
theory of the structure of spacetime
replacing Newtonian Mechanics.

Spacetime - is 3+1 4D set space(3) + time(1)

Event - point in spacetime

Path of a particle is a curve through spacetime worldline - parameterized 1-D set of events

Note time + space are treated differently
a particle can move freely through space
but only forward in time

Newton - spacetime involves "all of space at a
fixed moment in time"

Simultaneity - 2 events happen at the same time,
this is poorly defined in SR

light cone - future + past cones define observable
spacetime

Speed of light - limit is SR but not Newtonian

in Relativity there is no prefered coordinate
system so $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + (\Delta y')^2$
the spacetime distance between events is the same

inertial frame - natural generalization to spacetime created
using local observations

spacetime interval - $(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$
can be > 0, $= 0$ or < 0
timelike, null-like or spacelike

Minkowski space - flat spacetime

conventions $\left\{ \begin{array}{l} \text{in cart coords } x^\mu; \quad x^0 = ct \\ \qquad\qquad\qquad\qquad\quad x^1 = x \\ \qquad\qquad\qquad\qquad\quad x^2 = y \\ \qquad\qquad\qquad\qquad\quad x^3 = z \\ \\ \qquad\qquad\qquad x^i = \quad x^1 = x \\ \qquad\qquad\qquad\qquad\quad x^2 = y \\ \qquad\qquad\qquad\qquad\quad x^3 = z \end{array} \right.$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

proper time $-(\Delta \tau)^2 = -\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = -(\Delta s)^2$

line element - $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

if $x^\mu \to x^\mu(\lambda)$ $\quad \Delta s = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda$

$$\Delta \tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda$$

Lorentz Transformations

- Describes rotations in spacetime that leave spacetime distance intervals invariant for different inertial frames

translations : shift coords

$$x^\mu \rightarrow x^{\mu'} = \delta^{\mu'}_{\mu}(x^\mu + a^\mu)$$

spatial rotations & boosts

$$x^\mu \rightarrow x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu \qquad \text{or} \qquad x' = \Lambda x$$

$$\Rightarrow (\Delta s)^2 = (\Delta x)^T \eta (\Delta x) = (\Delta x')^T \eta (\Delta x')$$
$$= (\Delta x)^T \Lambda^T \eta \Lambda (\Delta x)$$

$$\therefore \eta = \Lambda^T \eta \Lambda$$

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_{\rho} \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\sigma} = \Lambda^{\mu'}{}_{\rho} \Lambda^{\nu'}{}_{\sigma} \eta_{\mu'\nu'}$$

Lorentz Group $SO(3,1)$

3×3 matrices

similar to the rotation group $SO(3)$

$R^T R = I$

special — det = 1

orthogonal — transvers is inverse

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\det \Lambda = 1$ much like $\det R = 1$

example $\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$ – rotation of x-y plane

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$ - boosts (t-x rotation)

note $\theta \in 0, 2\pi \quad \phi \in -\infty, \infty$

Lorentz transformations don't commute so they are nonabelian

Lorentz + translation form the 10 parameter nonabelian Poincare' group

$\therefore$ Lorentz transformations can be written as

$$\left.\begin{aligned} t' &= t\cosh\phi - x\sinh\phi \\ x' &= -t\sinh\phi + x\cosh\phi \end{aligned}\right] \text{ boosts}$$

$$\therefore \quad v = \frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi$$

rapidity $\rightarrow \phi = \tanh^{-1} v$

now $t' = \gamma(t - vx)$ $\gamma = 1/\sqrt{1-v^2}$

$x' = \gamma(x - vt)$

Vectors

- each 4-vector is located at a different pt in spacetime

tangent space at p or T_p - at each point p in spacetime we associate the set of all possible vectors located at the point

- see Fig 1.8 on page 16

vector field - set of vectors w/ exactly one at each pt in spacetime

tangent bundle $T(M)$ - set of all the tangent spaces of an n-dimensional manifold M which are assembled into a 2n dimensional manifold

basis - set of vectors which spans the vector space and is linearly independent

dimension - corresponds to the no. of basis vectors

Any vector can be written as a linear combo of basis vectors

$$\vec{A} = \underbrace{A^{\mu}}_{\text{components}} \overbrace{\hat{e}_{(\mu)}}^{\text{basis}}$$

tangent to curve $x^\mu(\lambda)$

$$V^\mu = \frac{dx^\mu}{d\lambda} \qquad V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$$

$$V = V^\mu \hat{e}_{(\mu)} = V^{\nu'} \hat{e}_{(\nu')} = \Lambda^{\nu'}{}_\mu V^\mu \hat{e}_{(\nu')}$$

$$\therefore \hat{e}_{(\mu)} = \Lambda^{\nu'}{}_\mu \hat{e}_{(\nu')}$$

also $\Lambda^\mu{}_{\nu'} \Lambda^{\nu'}{}_\rho = \delta^\mu_\rho$

Dual Vectors (One-Forms)

dual vector space - space of all linear maps from the original vector space to the real numbers

tangent space T_p, cotangent space T_p^*

if $\omega \in T_p^*$

scalars

$$\omega(a\vec{V} + b\vec{W}) = a\omega(\vec{V}) + b\omega(\vec{W}) \in R$$

dual vectors real no.

$$(a\omega + b\eta)(\vec{V}) = a\omega(\vec{V}) + b\eta(\vec{V})$$

basis dual vectors $\hat{\theta}^{(\nu)}$ where $\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^\nu_\mu$

$$\omega = \omega_\mu \hat{\theta}^{(\mu)}$$

$V^a \leftarrow$ contravariant $\qquad V_b \leftarrow$ covarient

$$\omega(V) = \omega_\mu \hat{\theta}^{(\mu)}(V^\nu \hat{e}_{(\nu)})$$

$$= \omega_\mu V^\nu \hat{\theta}^{(\mu)}(\hat{e}_{(\nu)})$$

$$= \omega_\mu V^\nu \delta^\mu_\nu$$

$$= \omega_\mu V^\mu \in R$$

$$V(\omega) = \omega(\vec{V}) = \omega_\mu V^\mu$$

$$\omega_{\mu'} = \Lambda^\nu_{\mu'} \omega_\nu$$

gradient (dual vector) ; $\quad d\phi = \frac{\partial \phi}{\partial x^\mu} \hat{\theta}^{(\mu)}$

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x_\mu} = \Lambda^\mu_{\mu'} \frac{\partial \phi}{\partial x^\mu}$$

shorthand $\quad \frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi = \phi_{,\mu}$

$$\partial_\mu \phi \frac{\partial x^\mu}{\partial \lambda} = \frac{d\phi}{d\lambda}$$